Expected Value, Certainty Equivalence, etc.

1 Expected value

Mathematically, an expected value calculation weighs each possible outcome by its likelihood, giving more weight to more likely outcomes and less weight to less likely outcomes. When there are a finite number of outcomes, calculate expected value by summing probability times value over all possible outcomes:

$$\text{Expected Value} \quad = \sum_{\text{Outcomes } i} \text{Probability}(i) \cdot \text{Value}(i).$$

The Greek letter $\sum$ ("sigma") is the mathematical notation for summation, e.g., $\sum_{y=1,2,3} y^2 = 1^2 + 2^2 + 3^2 = 14$. If you win \$1 if a coin comes up heads (H) and lose \$1 if it comes up tails (T), the expected value is

$$\begin{aligned} EV &= \Pr(H) \cdot (\$1) + \Pr(T) \cdot (\$-1) \\ &= \frac{1}{2} \cdot (\$1) + \frac{1}{2} \cdot (\$-1) \\ &= \$0. \end{aligned}$$

Infinite number of outcomes

Note: You are not expected to know this material; but it might be interesting. . . .

With an infinite number of outcomes (say, picking a random number between 0 and 1 and winning that amount of money), we can express the concept of expected value using **cumulative distribution functions (CDFs)** and **probability density functions (PDFs)**. The first idea here is to map all the possible outcomes onto the real number line. (This is easy in some cases, as with our example, where we can associate the outcome "0.55" with the point 0.55 on the number line.) The next idea is to consider the probability that the outcome is associated with a number less than or equal to x: we will call this CDF(x). As with all CDFs, the one in our example is a (weakly) increasing function with a minimum value of 0 (in our case, CDF(0) = 0) and a maximum value of 1 (in our case, CDF(1) = 1). For our example, $\text{CDF}(x) = x$ for $0 \le x \le 1$ and $\text{CDF}(x) = 1$ for $x \ge 1$. Finally, we take a derivative of CDF(x) to get PDF(x); the PDF, in other words, measures the rate of change of the CDF. This PDF(x) is the closest we can come to the intuition behind the finite-outcomes case: following the fundamental theorem of calculus, we know that

$$\int_{x_1}^{x_2} \text{PDF}(x) dx = \text{CDF}(x_2) - \text{CDF}(x_1),$$

i.e., that the area under the PDF between two points x_1 and x_2 tells us the probability of getting an outcome associated with a number between x_1 and x_2. (It follows for all PDFs that $\text{PDF}(x) \geq 0$ for all x and that $\int_{-\infty}^{\infty} \text{PDF}(x)dx = \text{CDF}(\infty) - \text{CDF}(-\infty) = 1 - 0 = 1$. In our specific example, we have $\text{PDF}(x) = 1$ for $0 \leq x \leq 1$ and $\text{PDF}(x) = 0$ otherwise.) **Finally**, we can define the **expected value** as

$$EV = \int_{-\infty}^{\infty} \text{Value}(x) \cdot \text{PDF}(x)dx.$$

In the example, we have $EV = \int_0^1 x \cdot (1)dx = 0.5$.

2 Certainty equivalence, take one

Consider an uncertain situation with two possible outcomes that depend on whether or not you get into an accident. Let w_1 (on the x-axis) measure your wealth if you don't have an accident and w_2 (on they y-axis) measure your wealth if you do have an accident. Assume that your initial endowment (i.e., without any insurance) is $w_1 = B$ and $w_2 = B - L$ (meaning that your base wealth is B and that you lose L in case of an accident) and that the probability of an accident is p (which implies that the probability of no-accident is $1 - p$). We can then consider all the usual stuff by treating w_1 and w_2 as different goods and writing your utility as $U(w_1, w_2)$. For example, we can determine an **indifference curve** of all points that give you the same utility as $(B, B - L)$. These points form the set

$$\{(w_1, w_2) : U(w_1, w_2) = U(B, B - L)\}.$$

One of these points is your **certainty equivalent wealth** w_{CE}, which is defined as the value that satisfies

$$U(w_{CE}, w_{CE}) = U(B, B - L).$$

In other words, you are indifferent between facing an uncertain situation (in which your wealth is B or $B - L$ depending on whether or not you get into an accident) and facing a certain situation of always having wealth w_{CE}, *regardless of whether or not you get into an accident.* [Side note: Be aware that the textbook defines certainty equivalence differently; we're going with the definition above, which is more common.]

Next, we can can draw a **fair odds line**, which has some elements in common with a budget constraint. This is the set of all points that give you the same *expected* wealth ("expected" in the sense of expected value) as your initial endowment point, i.e., the set

$$\{(w_1, w_2) : (1 - p)w_1 + (p)w_2 = (1 - p)B + (p)(B - L)\}.$$

In a competitive market with costless provision of insurance (e.g., no administrative costs), competition between insurance companies should drive profits down to zero, at which point you will be able to costlessly move along the fair odds line, the points of which are also known as **actuarially fair** contracts. In particular, you can move to the point (w_1, w_2) on the fair odds line with $w_1 = w_2$, i.e., to the point where $w_1 = w_2 = (1-p)B + (p)(B-L)$. This point corresponds with your **expected wealth** $E(w)$, and we can then calculate your **risk premium** as $E(w) - w_{CE}$; this risk premium measures the maximum amount *beyond actuarial fairness* that you'd be willing to pay to get full insurance: if you pay that extra amount for full insurance, you will end up with wealth w_{CE} regardless of whether or not you get into an accident, and that puts you back on your original indifference curve. If you're asked to pay any more for full insurance, you'll be better off declining; if you're offered insurance for less, you'll be better off. (This latter outcome is what happens when there's competition and costless provision of insurance, because in this case you pay nothing more than what is actuarially fair for full insurance.)

3 VNM utility functions

Recall that the assumptions made by von Neumann and Morgenstern allow us to write

$$U(w_1, w_2) = (1-p)u(w_1) + (p)u(w_2),$$

where $u(\cdot)$ is a a VNM utility function. In this case we can graph w on the x-axis and $u(w)$ on the y-axis. It turns out that a risk-averse individual will have a VNM utility function that is concave, i.e., that has a negative second derivative: $u''(x) < 0$. If you draw such a function (e.g., $u(w) = \sqrt{w}$), you can label points B and $B-L$ on the x-axis such that $B-L < B$, and then connect them with the points $u(B)$ and $u(B-L)$ on the y-axis. (You should have $u(B) > u(B-L)$.)

Next: We will now show that

$$u(E(w)) > E(u(w)),$$

i.e., that the individual would purchase costless insurance if it is offered. Costless insurance means that the individual could buy a contract ensuring her a wealth level of

$$E(w) = (1-p)B + (p)(B-L),$$

and consequently a utility level of

$$u(E(w)) = u([1-p]B + p[B-L]),$$

regardless of whether or not she gets into an accident. The inequality above shows that she prefers this to facing uncertainty and having expected utility

$$E(u(w)) = (1-p)u(B) + (p)u(B-L).$$

To see why this inequality holds, it is easiest to assume that $p = 0.5$, in which case $E(w)$ is a point halfway between $B-L$ and B on the x-axis and

$E(u(w))$ is a point halfway between $u(B - L)$ and $u(B)$ on the y-axis. The point on the y-axis corresponding to $E(w)$ is $u(E(w))$, and you should find that this point lies above $E(u(w))$. (Mathematically, this result follows from **Jensen's inequality**, which says that $u(E(w)) > E(u(w)) \iff u''(w) < 0$.) You can also find the point on the x-axis that corresponds to $E(u(w))$; this is the individual's **certainty equivalent wealth** w_{CE}: having this level of wealth regardless of whether or not you get into an accident gives you the same expected utility as your original (uncertain) endowment of $(B, B - L)$. As before, the **risk premium** $E(w) - w_{CE}$ measures the maximum amount you'd be willing to pay for full insurance.

4 One case of the optimality of full insurance

Consider a risk-averse individual, i.e., one with a VNM utility function that is concave: $u''(w) < 0$. (This means that $u'(w)$ is strictly decreasing; one implication that will come in handy later is that $u'(w_1) = u'(w_2) \iff w_1 = w_2$.) This individual has initial wealth of B, a probability p of getting into an accident that will result in a loss L, and the ability to buy any insurance contract that is actuarially fair. What kind of contract will she buy?

First consider the limitations imposed by actuarial fairness: if the individual wants to receive an insurance payment of G in the event of an accident, she must pay a **premium** (e.g., a monthly payment, one that is paid regardless of whether or not an accident occurs) of z such that

$$(1 - p)z + (p)(z - G) = 0.$$

This means that the insurance company (which we're assuming has no administrative or other costs) is making zero profits, and it reduces to

$$z = pG.$$

In words, the premium must equal the expected pay-out from the insurance policy, where "expected" is again used in the sense of expected value.

So: the individual now faces the problem of choosing z and G to maximize

$$E(u(w)) = (1 - p) \cdot u(B - z) + p \cdot u(B - z - L + G)$$

subject to the constraint $z = pG$. Substituting the constraint into the objective function, we can rewrite the problem as choosing G to maximize

$$E(u(w)) = (1 - p) \cdot u(B - pG) + p \cdot u(B - pG - L + G).$$

We maximize this by taking a derivative with respect to the choice variable G and setting it equal to zero. Using the chain rule we get

$$(1 - p) \cdot u'(B - pG) \cdot (-p) + p \cdot u'(B - pG - L + G) \cdot (1 - p) = 0.$$

Cancel out the term $(1-p) \cdot p$ and rearrange to get

$$u'(B - pG) = u'(B - pG - L + G).$$

As argued at the top of this section, the only way $u'(w_1) = u'(w_2)$ is if $w_1 = w_2$, i.e., if

$$B - pG = B - pG - L + G.$$

This simplifies to $G = L$. In words: the optimizing individual chooses to receive compensation $G = L$ in the event of an accident, meaning that her wealth if there is an accident is $B-z-L+G = B-z = B-pG$, which is exactly the same as her wealth if there is no accident. Translation: this optimizing individual will buy full insurance.