Answer Key to Problem Set #2: Expected Value and Insurance

1. (a) We have $u'(w) = \frac{1}{2}w^{-\frac{1}{2}}$, so $u''(w) = -\frac{1}{4}w^{-\frac{3}{2}}$. As we will see below, $u''(w) < 0$ indicates that the individual is risk-averse.

   (b) The expected amount of money he will lose is

   $$(.2)(\$300) + (.8)(0) = \$60.$$

   His expected wealth is

   $$(.2)(\$100) + (.8)(\$400) = \$340.$$

   (c) His expected utility is

   $$(.2) \cdot u(\$100) + (.8) \cdot u(\$400) = 2 + 16 = 18.$$

   (d) His certainty equivalent wealth is the certain wealth w_{CE} that gives him the same expected utility as the uncertain certain he starts out it, i.e., the certain wealth w_{CE} that gives him an expected utility of 18. Solving $u(w_{CE}) = 18$ for w_{CE} gives us $w_{CE} = \$324$.

   (e) The maximum amount he would pay for full insurance is his initial wealth minus his certainty equivalent wealth:

   $$\$400 - \$324 = \$76.$$

   The story here is basically this: He starts out with base wealth of \$400. If he pays \$76 to the insurance company in exchange for them covering his loses in case of an accident, his wealth (regardless of whether or not an accident occurs) will always be

   $$\$400 - \$76 = \$324.$$

   From above we know that he is indifferent between having certain wealth of \$324 and facing the uncertain situation he started out with: both give him an expected utility of 18.

   So: if he pays \$76 for full insurance, he ends up with the same expected utility that he started out with. If he can get full insurance for one dollar less (i.e., for \$75), buying the insurance will *increase* his expected utility, so he will want to buy the insurance. If he can only get insurance by paying one dollar more (i.e., for \$77), buying the insurance will *decrease* his expected utility, so he will refuse to buy the insurance.

(f) He's willing to pay a maximum of \$76 for full insurance, and his expected loss is \$60. So his risk premium is \$16. This makes sense because he is risk-neutral. You can also think of the risk premium as his *consumer surplus* if he is able to buy actuarially fair insurance (i.e., if he is able to buy full insurance for an amount equal to his expected loss). Just as a consumer who is willing to pay up to \$21 for a hat and ends up paying \$10 for it gains \$11 in consumer surplus, a risk-averse individual is made better off by purchasing an actuarially fair insurance policy.

(g) The second derivative is now $u''(w) = -\frac{3}{4}w^{-\frac{3}{2}}$, which is of the same sign as before but three times larger in magnitude. His expected loss and expected wealth are unchanged at \$60 and \$340, respectively. His expected utility is now

$$(.2) \cdot u(\$100) + (.8) \cdot u(\$400) = 10 + 64 = 74$$

instead of 18. To get his certainty equivalent wealth, solve $u(w_{CE}) = 74$ for w_{CE} to get $w_{CE} = \$324$, which is the same as before. The maximum amount he would pay for full insurance is therefore also the same as before ($\$400 - \$324 = \$76$); so is his risk premium ($\$76 - \$60 = \$16$). The point here is that nothing *observable* changes: it is impossible to distinguish between an individual maximizing $u(w) = \sqrt{w}$ and an individual maximizing $u(w) = 3\sqrt{w} + 20$. These individuals behave identically in terms of what kinds of insurance policies they will or will not buy, which is exactly what it means to say that VNM utility functions are unique only up to affine transformations: if $u(w)$ represents an individual's preferences, then so will $v(w) = a + bu(w)$ for any constants a and b with $b > 0$. (Note that we could tell these individuals apart if we could measure utility directly, e.g., with a "psychogalvanometer". But—although economists wouldn't object in any way if utility did turn out to be directly measurable—economic theory does not *need* for utility to be an actual measurable thing, or even for it to be actual in any way: utility functions are simply handy ways of representing an individual's preferences, and the same preferences can be represented by many different utility functions.)

2. (a) We have $u'(w) = 20$, so $u''(w) = 0$. As we will see below, $u''(w) = 0$ indicates that the individual is risk-neutral.

(b) The expected amount of money he will lose is

$$(.2)(\$300) + (.8)(0) = \$60.$$

His expected wealth is

$$(.2)(\$100) + (.8)(\$400) = \$340.$$

(c) His expected utility is

$$(.2) \cdot u(\$100) + (.8) \cdot u(\$400) = 402 + 6,408 = 6,810.$$

(d) His certainty equivalent wealth is the certain wealth w_{CE} that gives him the same expected utility as the uncertain certain he starts out it, i.e., the certain wealth w_{CE} that gives him an expected utility of 5,210. Solving $u(w_{CE}) = 6,810$ for w_{CE} gives us $w_{CE} = \$340$.

(e) The maximum amount he would pay for full insurance is his initial wealth minus his certainty equivalent wealth:

$$\$400 - \$340 = \$60.$$

The story here is basically this: He starts out with base wealth of \$400. If he pays \$60 to the insurance company in exchange for them covering his loses in case of an accident, his wealth (regardless of whether or not an accident occurs) will always be

$$\$400 - \$60 = \$340.$$

From above we know that he is indifferent between having certain wealth of \$340 and facing the uncertain situation he started out with: both give him an expected utility of 6,810.

So: if he pays \$60 for full insurance, he ends up with the same expected utility that he started out with. If he can get full insurance for one dollar less (i.e., for \$59), buying the insurance will *increase* his expected utility, so he will want to buy the insurance. If he can only get insurance by paying one dollar more (i.e., for \$61), buying the insurance will *decrease* his expected utility, so he will refuse to buy the insurance.

(f) He's willing to pay a maximum of \$60 for full insurance, and his expected loss is \$60. So his risk premium is \$0. This makes sense because he is risk-neutral. You can also think of the risk premium as his *consumer surplus* if he is able to buy actuarially fair insurance (i.e., if he is able to buy full insurance for an amount equal to his expected loss). Just as a consumer who is willing to pay up to \$10 for a hat and ends up paying \$10 for it gains no consumer surplus, a risk-neutral individual gains no benefit from an actuarially fair insurance policy.

3. (a) We have $u'(w) = 2w$, so $u''(w) = 2$. As we will see below, $u''(w) > 0$ indicates that the individual is risk-loving.

(b) The expected amount of money he will lose is

$$(.2)(\$300) + (.8)(0) = \$60.$$

His expected wealth is

$$(.2)(\$100) + (.8)(\$400) = \$340.$$

(c) His expected utility is

$$(.2) \cdot u(\$100) + (.8) \cdot u(\$400) = 130,000.$$

(d) His certainty equivalent wealth is the certain wealth w_{CE} that gives him the same expected utility as the uncertain certain he starts out it, i.e., the certain wealth w_{CE} that gives him an expected utility of 130,000. Solving $u(w_{CE}) = 130,000$ for w_{CE} gives us $w_{CE} \approx \$361$.

(e) The maximum amount he would pay for full insurance is his initial wealth minus his certainty equivalent wealth:

$$\$400 - \$361 = \$39.$$

The story here is basically this: He starts out with base wealth of \$400. If he pays \$39 to the insurance company in exchange for them covering his loses in case of an accident, his wealth (regardless of whether or not an accident occurs) will always be

$$\$400 - \$39 = \$361.$$

From above we know that he is indifferent between having certain wealth of \$361 and facing the uncertain situation he started out with: both give him an expected utility of 130,000.

So: if he pays \$39 for full insurance, he ends up with the same expected utility that he started out with. If he can get full insurance for one dollar less (i.e., for \$38), buying the insurance will *increase* his expected utility, so he will want to buy the insurance. If he can only get insurance by paying one dollar more (i.e., for \$40), buying the insurance will *decrease* his expected utility, so he will refuse to buy the insurance.

(f) He's willing to pay a maximum of \$39 for full insurance, and his expected loss is \$60. So his risk premium is $-\$21$. This makes sense because he is risk-loving. You can also think of the risk premium as his *consumer surplus* if he is able to buy actuarially fair insurance (i.e., if he is able to buy full insurance for an amount equal to his expected loss). Just as a consumer who is willing to pay up to \$5 for a hat and ends up paying \$10 for it loses \$5 in consumer surplus, a risk-loving individual is made worse off by purchasing an actuarially fair insurance policy.

4. (a) See the figures on the next page, which assume $p = .5$. In general, the equation for the indifference curve is

$$\{(w_1, w2) : p\sqrt{w_2} + (1-p)\sqrt{w_1} = p\sqrt{100} + (1-p)\sqrt{400}\}.$$

The equation for the fair-odds line with actuarially fair insurance is

$$\{(w_1, w2) : pw_2 + (1-p)w_1 = p \cdot 100 + (1-p) \cdot 400\}.$$

The equation for the fair-odds line with the $8 application fee is

$$\{(w_1, w2) : pw_2 + (1-p)w_1 = p \cdot 100 + (1-p) \cdot 400 - 8\}.$$

Note that the fair-odds line with actuarially fair insurance intersects the 45 degree "certainty equivalence" line at $(400-300p, 400-300p)$, and that the fair-odds line with the $8 application fee intersects the 45 degree line at $(400 - 300p - 8, 400 - 300p - 8)$. This (hopefully) helps to show that the latter line is $8 below plus $8 to the left of the former line.

(b) If she buys insurance, she has to pay $pL + 8 = 300p + 8$ to cover her expected loss plus the application fee. So if she buys insurance she will, with certainty, have wealth $400 - (pL + 8) = 392 - 300p$. Her expected utility from this level of wealth is $\sqrt{392 - 300p}$. Her expected utility if she doesn't buy insurance is $p\sqrt{100}+(1-p)\sqrt{400} = 20-10p$. Setting these terms equal to each other will generate critical values where she is indifferent between buying insurance and not buying insurance:

$$\sqrt{392 - 300p} = 20 - 10p \iff 392 - 300p = 400 - 400p + 100p^2.$$

This simplifies to $p^2 - p + .08 = 0$, and solving using the quadratic formula gives

$$p = \frac{1}{2}\left(1 \pm \sqrt{1 - .32}\right) = \frac{1}{2}(1 \pm .8) \Longrightarrow p \in \{.1, .9\}.$$

Plugging values on either side of and in between these critical values (e.g., $p = 0, p = 1, p = .5$) into her expected utility with insurance ($\sqrt{392 - 300p}$) and her expected utility without insurance ($20 - 10p$) and comparing them confirms that she will buy insurance for $.1 < p < .9$ but will not buy insurance for $p < .1$ or $p > .9$. (When $p = .1$ or $p = .9$, she is indifferent between buying insurance and self-insuring.) Intuitively, she will not buy insurance for $p < .1$ because the odds of getting into an accident are so small that it's not worth paying $8 more than her expected loss for insurance; and she will not buy insurance for $p > .9$ because the odds of *not* getting into an accident are so small that it's not worth paying $8 more than her expected loss for insurance. (Note that insurance will *always* cost her at least her expected loss. What insurance protects her against is not the *expected value* associated with her loss but the *variance* associated with her loss, as described in the next problem)

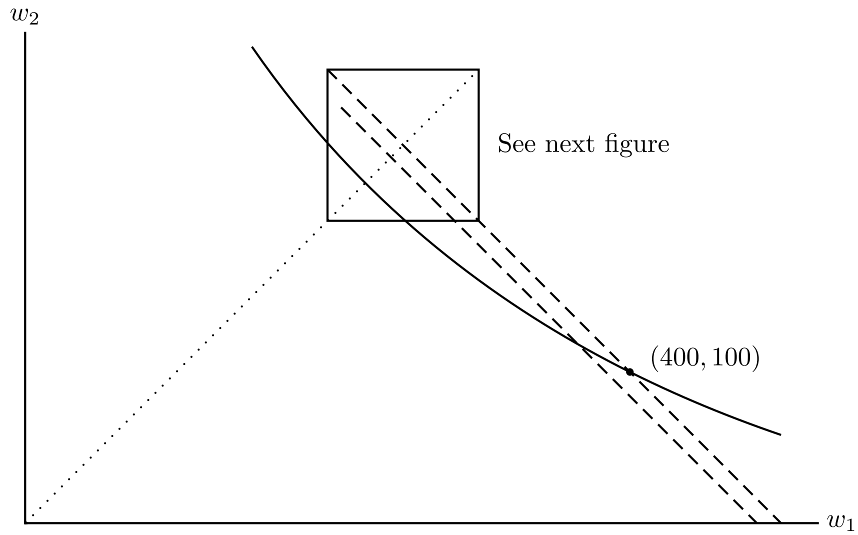

Figure 1: Solid line is indifference curve; dashed lines are fair-odds lines; dotted line is certainty equivalence line. This figure assumes $p = .5$.

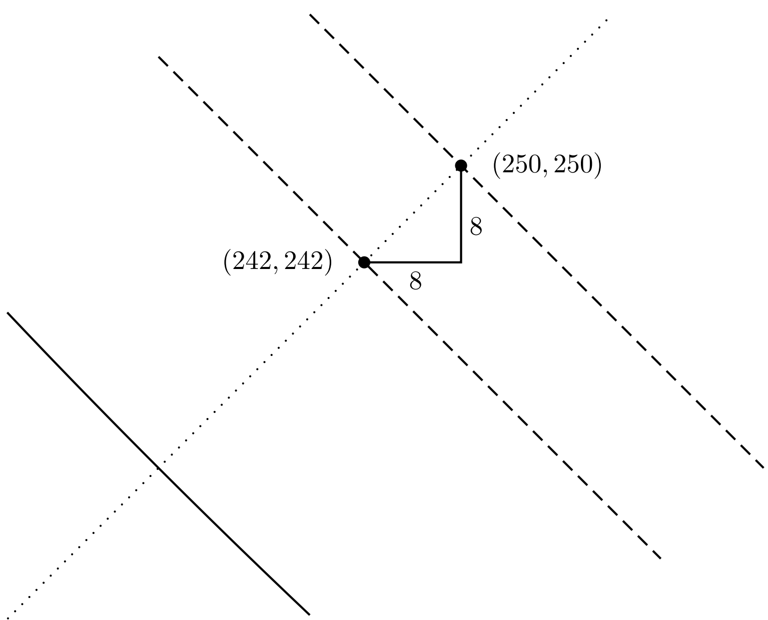

Figure 2: An enlargement of the previous picture, showing how the $8 application fee affects the fair-odds line. Again, this figure assumes $p = .5$.

(c) The variance of her loss is

$$p\left[(300-300p)^2\right]+(1-p)\left[(0-300p)^2\right]=900000p(1-p).$$

The variance of her wealth is

$$p\left[(100-(400-300p))^2\right]+(1-p)\left[(400-(400-300p))^2\right],$$

which simplifies to the same thing. Notice that these variances are quadratic functions that have a maximum value when $p = .5$ and have a value of 0 when $p = 0$ or $p = 1$. The fact that she doesn't buy actuarially unfair insurance (i.e., insurance with an \$8 application fee) when p is close to 0 or 1 is connected to the fact that the variance of her loss (or of her wealth) is small when p is close to 0 or 1.

(d) Again, nothing measurable changes. The values of p for which she is indifferent between purchasing insurance and self-insuring again satisfy

$$u(392-300p)=p\cdot u(100)+(1-p)\cdot u(400),$$

and if $u(w) = 3\sqrt{w} + 20$ this still simplifies to $p^2 - p + .08 = 0$ and we get all of the same answers.