Answer Key for Part I: The Optimizing Individual

Chapter 1: Decision Theory

1. You should not believe this argument because the amount spent on the rights to the Olympic Games is a sunk cost. The reason the network showed the Olympics for so many hours was because that was the decision that maximized profits.

   We can also see this mathematically: the amount spent on the rights to the Games is a constant term in the profit function, e.g., $\pi = f(h) - c$ where $f(h)$ are advertising revenues from showing the games for h hours and c is the amount spent on the rights to the games. To maximize this, we take a derivative with respect to h and set it equal to zero. Note that the constant term c drops out when we take a derivative, and therefore has no impact on the optimal choice of h.

2. A billionaire would probably take this bet because its expected value is \$4.5 million. A penniless person would probably not take the bet, because the risk of ending up with zero is too great. So sunk costs cannot be entirely ignored in decision-making; rather, the point is that it is not possible to base decisions *solely* on sunk costs.

3. The law does not deter Alice from committing additional crimes because she's already facing the death penalty if she's caught. The law does deter Betty, because she hasn't killed anybody yet.

4. You should not believe the spokesman's explanation because the R&D expenditure is a sunk cost. If it spent twice as much or half as much to discover the drug, it should still charge the same price, because that's the price that maximizes profit. And that's the alternative explanation for why the drug company charges such a high price: that's the price that maximizes profit.

Calculus Problems

C-1. Microeconomics is about the actions and interactions of optimizing agents (e.g., profit-maximizing firms, utility-maximizing consumers). For differentiable functions with interior maxima or minima, the way to find those interior maxima or minima is to take a derivative and set it equal to zero. This gives you *candidate values* for maxima or minima; the reason is that slopes (i.e., derivatives) are equal to zero at the top of a hill (a maximum) or at the bottom of a valley (a minimum).

C-2.

$$\begin{aligned} \frac{d}{dx}[(c \cdot f(x)] &= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\ &= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \cdot \frac{d}{dx}[f'(x)]. \end{aligned}$$

C-3. $\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x) = 2x.$

For higher powers, use induction: We've just proved that our rule ($\frac{d}{dx}(x^n) = nx^{n-1}$) is true for $n = 2$. So now we assume that it's true for n ($\frac{d}{dx}(x^n) = nx^{n-1}$) and need to show that it's true for x^{n+1}. But we can just use the same trick again:

$\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x \cdot x^n) = x \cdot \frac{d}{dx}(x^n) + x^n \cdot \frac{d}{dx}(x) = xnx^{n-1} + x^n = (n+1)x^n.$

C-4.

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{d}{dx}\left[f(x) \cdot (g(x))^{-1}\right] \\ &= f(x) \cdot \frac{d}{dx}\left[(g(x))^{-1}\right] + (g(x))^{-1} \cdot \frac{d}{dx}[f(x)] \\ &= f(x) \cdot (-1)[g(x)]^{-2}\frac{d}{dx}[g(x)] + (g(x))^{-1} \cdot \frac{d}{dx}[f(x)] \\ &= \frac{-f(x) \cdot g'(x)}{[g(x)]^2} + \frac{f'(x)}{g(x)} \cdot \frac{g(x)}{g(x)} \\ &= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}. \end{aligned}$$

C-5. (a) We have $f'(x) = 2x$ and $f''(x) = 2$. Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0) = 2 > 0$, identifies this as a minimum; this is also clear from a graph of $f(x)$.

(b) We have $f'(x) = 2(x^2+2) \cdot \frac{d}{dx}(x^2+2) = 4x(x^2+2) = 4x^3+8x$ and $f''(x) = 12x^2+8$. Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. (We are not interested in imaginary roots such as $i\sqrt{2}$.) Note that the sign of the second derivative, $f''(0) = 8 > 0$, identifies this as a minimum.

(c) We have $f'(x) = \frac{1}{2} \cdot (x^2+2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2+2) = x \cdot (x^2+2)^{-\frac{1}{2}}$ and

$$\begin{aligned} f''(x) &= x \cdot \frac{d}{dx}\left[(x^2+2)^{-\frac{1}{2}}\right] + \left[(x^2+2)^{-\frac{1}{2}}\right] \cdot \frac{d}{dx}(x) \\ &= x \cdot \left(-\frac{1}{2}\right) \cdot (x^2+2)^{-\frac{3}{2}} \frac{d}{dx}(x^2+2) + \left[(x^2+2)^{-\frac{1}{2}}\right] \cdot 1 \\ &= -x^2 \cdot (x^2+2)^{-\frac{3}{2}} + (x^2+2)^{-\frac{1}{2}} \end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0) = \frac{1}{\sqrt{2}} > 0$, identifies this as a minimum.

(d) We have

$$\begin{aligned} f'(x) &= -x \cdot \frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right] + \left[(x^2+2)^{\frac{1}{2}}\right] \cdot \frac{d}{dx}(-x) \\ &= -x \cdot \left(\frac{1}{2}\right) \cdot (x^2+2)^{-\frac{1}{2}} \frac{d}{dx}(x^2+2) + \left[(x^2+2)^{\frac{1}{2}}\right] \cdot (-1) \\ &= -x^2 \cdot (x^2+2)^{-\frac{1}{2}} - (x^2+2)^{\frac{1}{2}} \\ \text{and } f''(x) &= \frac{d}{dx}\left[-x^2 \cdot (x^2+2)^{-\frac{1}{2}}\right] - \frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right] \\ &= -x^2 \cdot \frac{d}{dx}\left[(x^2+2)^{-\frac{1}{2}}\right] + (x^2+2)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(-x^2\right) \\ &\quad -\frac{1}{2}\left[(x^2+2)^{-\frac{1}{2}}\right] \frac{d}{dx}(x^2+2) \\ &= -x^2 \cdot \left(-\frac{1}{2}\right)\left[(x^2+2)^{-\frac{3}{2}}\right] \frac{d}{dx}(x^2+2) - 2x(x^2+2)^{-\frac{1}{2}} \\ &\quad -x(x^2+2)^{-\frac{1}{2}} \\ &= x^3(x^2+2)^{-\frac{3}{2}} - 3x(x^2+2)^{-\frac{1}{2}} \end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x) = 0$. Multiplying both sides by $(x^2+2)^{\frac{1}{2}}$ we get $-x^2 - (x^2+2) = 0$, which simplifies to $x^2 = -1$. Since this equation has no solutions, $f(x)$ has no interior maxima or minima.

(e) We have

$$f'(x) = \frac{1}{(x^2+2)^{\frac{1}{2}}} \cdot \frac{d}{dx}\left[(x^2+2)^{\frac{1}{2}}\right]$$

$$\begin{aligned}
&= (x^2+2)^{-\frac{1}{2}} \cdot x \cdot (x^2+2)^{-\frac{1}{2}} \text{(from (c))} \\
&= x \cdot (x^2+2)^{-1} \\
\text{and } f''(x) &= x \cdot \frac{d}{dx}\left[(x^2+2)^{-1}\right] + \left[(x^2+2)^{-1}\right]\frac{d}{dx}(x) \\
&= x(-1)(x^2+2)^{-2}\frac{d}{dx}(x^2+2) + (x^2+2)^{-1} \\
&= -2x^2(x^2+2)^{-2} + (x^2+2)^{-1}
\end{aligned}$$

Candidate solutions for interior maxima or minima are where $f'(x) = 0$. The only candidate is $x = 0$, which turns out to be a minimum value of the function $f(x)$. Note that the sign of the second derivative, $f''(0) = \frac{1}{2} > 0$, identifies this as a minimum.

C-6. (a) We have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(-3x) + \frac{\partial}{\partial x}(2y) \\
&= y \cdot \frac{\partial}{\partial x}(x^2) - 3 \cdot \frac{\partial}{\partial x}(x) + 0 \\
&= 2xy - 3 \\
\text{and } \frac{\partial}{\partial y} &= \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(-3x) + \frac{\partial}{\partial y}(2y) \\
&= x^2 \cdot \frac{\partial}{\partial y}(y) + 0 + 2 \cdot \frac{\partial}{\partial y}(y) \\
&= x^2 + 2
\end{aligned}$$

(b) We have $\frac{\partial}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) = ye^{xy}$. Since $f(x, y)$ is symmetric in x and y, we must also have $\frac{\partial}{\partial y} = xe^{xy}$.

(c) We have $\frac{\partial}{\partial x} = y^2 \cdot \frac{\partial}{\partial x}(e^x) - 0 = y^2e^x$ and $\frac{\partial}{\partial x} = e^x \cdot \frac{\partial}{\partial x}(y^2) - 2 = 2ye^x - 2$.

C-7. (a) The slope of the demand curve is $\frac{dq}{dp} = -2 < 0$, which makes sense because as the price goes up consumers should want to buy less.

(b) The inverse demand curve is $p = 10 - .5q$.

(c) The slope of the inverse demand curve is $-.5$, which is the inverse of the slope of the demand curve.

C-8. (a) The inverse demand curve is $p = 20 - q$, and substituting into the profit function yields $\pi = (20 - q)q - 2q - F = 18q - q^2 - F$. Taking a derivative and setting it equal to zero gives us our candidate solution for an interior maximum: $\pi' = 0 \Longrightarrow 18 - 2q = 0 \Longrightarrow q^* = 9$. Substituting this back into the inverse demand curve yields $p^* = 11$, so that the profit-maximizing profit level is $\pi^* = 11 \cdot 9 - 2 \cdot 9 - F = 81 - F$.

(b) We can see from above that the monopolist will choose not to enter for $F > 81$: zero profits are better than negative profits. Note that we would get a **corner solution** to the maximization problem in this case. The answer $q^* = pi^* = 0$ does *not* show up as one of our candidate interior solutions.

C-9. In a competitive market the firm is assumed to be so small that its choice of q doesn't affect the market price p. So the firm treats p like a constant and takes a derivative of the profit function to find candidate (interior) maxima: $\pi' = 0 \Longrightarrow p - C'(q) = 0 \Longrightarrow p = C'(q^*)$. This says that the firm should produce until price equals marginal cost. Give $C(q)$ as above, we get $p = q + 2 \Longrightarrow q^* = p - 2$.

C-10. If the monopolist chooses to produce q, the inverse demand curve establishes the maximum price as $p(q)$. Substituting this into the profit function gives $\pi(p, q) = pq - C(q) \Longrightarrow \pi(q) = p(q) \cdot q - C(q)$. Taking a derivative and setting it equal to zero to find candidate (interior) maxima yields $p(q) \cdot 1 + p'(q) \cdot q - C'(q) = 0$. Given the specific demand and cost curves above, we get $(20 - q) + (-1)q - (q + 2) = 0 \Longrightarrow q^* = 6$.

C-11. First simplify a bit: $U = p_B^{-\frac{1}{2}} \cdot (100Z - p_Z Z^2)^{\frac{1}{2}}$. Then take a derivative with respect to Z:

$$
\begin{aligned}
\frac{d}{dZ}U &= p_B^{-\frac{1}{2}} \cdot \frac{d}{dZ}\left[(100Z - p_Z Z^2)^{\frac{1}{2}}\right] \\
&= p_B^{-\frac{1}{2}} \cdot \frac{1}{2}(100Z - p_Z Z^2)^{-\frac{1}{2}} \frac{d}{dZ}(100Z - p_Z Z^2) \\
&= p_B^{-\frac{1}{2}} \cdot \frac{1}{2}(100Z - p_Z Z^2)^{-\frac{1}{2}} \cdot (100 - 2p_Z Z).
\end{aligned}
$$

Setting this derivative equal to zero we find that the only candidate solution is $100 - 2p_z Z = 0 \Longrightarrow Z^* = \frac{50}{p_Z}$.

C-12. (a) Firm 1 chooses q_1 to maximize $\pi_1 = (20 - q_1 - q_2)q_1 - 2q_1 = 18q_1 - q_1^2 - q_1 q_2$.

(b) Symmetrically, Firm 2 chooses q_2 to maximize $\pi_2 = (20 - q_1 - q_2)q_2 - 2q_2 = 18q_2 - q_2^2 - q_1 q_2$.

(c) The difference between the two maximization problems is that q_1 is a constant in Firm 2's problem (because Firm 1 has already chosen q_1 by the time Firm 2 gets to pick q_2) but q_2 is *not* a constant in Firm 1's problem (because Firm 1's choice of q_1 will probably affect Firm 2's choice of q_2).

(d) Take a partial derivative of Firm 2's profit function with respect to its choice variable and set it equal to zero: $\frac{\partial}{\partial q_2}\pi_2 = 18 - 2q_2 - q_1 = 0 \Longrightarrow q_2^* = 9 - .5q_1$.

(e) Firm 1 needs to anticipate how its choice of q_1 will affect Firm 2's behavior in order to choose q_1 optimally.

(f) Plug Firm 2's best response function into Firm 1's objective function, take a derivative, and set it equal to zero to find Firm 1's profit-maximizing choice of q_1:

$$\begin{aligned}\frac{\partial}{\partial q_1}\pi_1 &= \frac{\partial}{\partial q_1}\left[18q_1 - q_1^2 - q_1(9 - .5q_1)\right] \\ &= 18 - 2q_1 - 9 + q_1 \\ &= 9 - q_1 \Longrightarrow q_1^* = 9.\end{aligned}$$

(g) Firm 1's optimal choice is $q_1^* = 9$, which means that Firms 2's optimal choice is $q_2^* = 9 - .5(9) = 4.5$, which means that $p = 20 - 9 - 4.5 = 6.5$, which means that $\pi_1^* = 6.5 \cdot 9 - 2 \cdot 9 = 40.5$ and $\pi_2^* = 6.5 \cdot 4.5 - 2 \cdot 4.5 = 20.25$. Firm 1 comes out ahead, so there is a first mover advantage here.

Chapter 2: Optimization and Risk

1. The expected value is $\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = \frac{21}{6}$.

2. The expected value is $\frac{1}{3}(99) + \frac{2}{3}(-33) = 11$.

3. You should switch: your odds of winning will increase from $\frac{1}{3}$ to $\frac{2}{3}$. A more extreme example may help provide some intuition behind this result: assume that there are 100 doors, only one of which leads to a car; after you pick a door, Monty opens up 98 of the other doors to reveal goats and then offers you the opportunity to switch to the remaining unopened door. Doing so will increase your odds of winning from $\frac{1}{100}$ to $\frac{99}{100}$.

   Still confused? Go online[1] for a fun computer simulation of the Monty Hall problem (with accompanying discussion).

4. (a) The expected value of guessing randomly is $\frac{1}{5}(1) + \frac{4}{5}(0) = \frac{1}{5}$.

   (b) If an incorrect answer is worth x, the expected value from guessing randomly is $\frac{1}{5}(1) + \frac{4}{5}(x) = \frac{1+4x}{5}$. If the teacher wants this expected value to equal zero, she must set $x = -\frac{1}{4}$.

   (c) Since this makes random guessing a fair bet, it will discourage risk averse students but not risk loving students.

   (d) If you can't eliminate any answers, the expected value of guessing randomly is $\frac{1}{5}(1) + \frac{4}{5}\left(-\frac{1}{3}\right) = -\frac{1}{15}$. If you can eliminate one answer, you have a 1 in 4 chance of getting the right answer if you guess randomly, so your expected value if you can eliminate one answer is $\frac{1}{4}(1) + \frac{3}{4}\left(-\frac{1}{3}\right) = 0$. If you can eliminate two answers, you have a 1 in 3 chance of getting the right answer if you guess randomly, so your

[1] http://cartalk.cars.com/About/Monty/

expected value if you can eliminate two answers is $\frac{1}{3}(1) + \frac{2}{3}\left(-\frac{1}{3}\right) = \frac{1}{9}$. So you need to eliminate at least two answers in order to make random guessing yield an expected value greater than zero.

5. No, they are not particularly risky. This is because of the **law of large numbers**, discussed in Section 2.1. The individual bettor plays roulette only a few times, and so faces a lot of risk. The casino plays roulette thousands of times each day, and so has a very good idea of what the overall outcome will be; since each \$1 wager has an expected payoff of only \$.95, it can expect to gain about \$.05 for every dollar wagered.

   Similarly, although insurance companies have no idea whether an individual driver is going to get into an accident this year, or whether an individual person is going to die this year, or whether an individual home is going to burn to the ground this year, the law of large numbers usually gives them a very good idea of the percentage of accidents or the percentage of deaths or the percentage of fires to expect from the hundreds of thousands of cars, lives, and homes they cover.

   So casinos or insurance companies are not necessarily any riskier as business endeavors than, say, running a photocopy shop.

6. Your expected value from Option 1 is .8(40)+.2(-20) = 28. Your expected value from Option 2 is .8(-5)+.2(100) = 16. Your expected value from Option 3 is 0. So Option 1 maximizes your expected value.

7. This is a difficult problem. For more information on it, read Raiffa's book or do some research on Bayes's Rule.

8. (a) Your expected value from bidding b in either type of auction is

$$\text{Prob}(b \text{ wins}) \cdot \text{Value}(b \text{ wins}) + \text{Prob}(b \text{ loses}) \cdot \text{Value}(b \text{ loses}).$$

   In a first-price auction, Value(b wins)$= 100-b$ and Value(b loses)$= 0$; so your expected value is

$$\text{Prob}(b \text{ wins}) \cdot (100 - b) + 0.$$

   (b) In a second-price auction, Value(b wins)$= 100 - c$, where c is the highest bid less than b, and Value(b loses)$= 0$. So your expected value is

$$\text{Prob}(b \text{ wins}) \cdot (100 - c) + 0.$$

   (c) Chapter 12 discusses this in more detail.

Chapter 3: Optimization over Time

1. (a) Plug \$100, 5%, and 30 years into the future value of a lump sum formula to get $y \approx \$432.19$.

   (b) Plug \$432.19, 5%, and 30 years into the present value of a lump sum formula to get $z \approx \$100$.

   (c) They are equal. The explanation here is that the formulas for present value and future value of lump sums are inverses of each other in that you can rearrange either equation to get the other:

$$\text{PV} = \frac{FV}{(1+s)^n} \Longleftrightarrow \text{FV} = (PV)(1+s)^n.$$

2. Intuitively, there shouldn't be much difference. Mathematically, the annuity formula approaches the perpetuity formulas as n approaches infinity:

$$\lim_{n\to\infty} x \left[\frac{1 - \dfrac{1}{(1+s)^n}}{s} \right] = \frac{x}{s}.$$

3. The perpetuity formula says that the present value of a perpetuity paying x every year is $\frac{x}{s}$. This is like living off the interest because if you put $\frac{x}{s}$ in the bank, every year you will get interest of $r \cdot \frac{x}{s} = x$, so with this principal you can finance annual consumption of x forever.

4. (a) Plug \$1 million, 5%, and 20 years into the annuity formula to get about \$12.5 million as the present value of the annuity.

   (b) Plug \$1 million and 5% into the perpetuity formula to get \$20 million as the present value of the perpetuity. Note that the extra payments you get—\$1 million annually beginning in year 21—are only worth about \$7.5 million in present value terms!

   (c) Increasing r will make the lump sum payment more attractive, and decreasing r will make the annual payments more attractive. Trial and error yields $s \approx .075$ as the interest rate that makes the two payoffs equal in present value terms.

   (d) The hard way to do this is to just calculate the present value of each payment and then add them all together. Easy way #1 is to realize that the difference between the end-of-year payments and the beginning-of-year payments is just an extra payment at the beginning of the first year and a lost payment at the end of the 20th year. The present value of \$1 million today is \$1 million, and the present value of \$1 million at the end of 20 years is \$380,000. Their difference is \$620,000, so adding this to the answer from (a) yields \$13.08 million.

Easy way #2 is to see that the answer from (a) is the right answer from the perspective of one year ago, so using the future value of a lump sum formula to push this answer one year into the future gives us \$12.46(1.05) = \$13.08 million.

5. (a) Note: The elements of this problem featuring logarithms are not fair game for the exam. Having said that: Solving $2x = x(1.10)^t$ using logarithms yields $t = 7.27$, pretty close to the Rule of 72 estimate of 7.2.

   (b) The Rule of 72 predicts 14.4 years at 5% and .72 years at 100%. The correct answer at 100% is 1 year (obviously, since the interest rate is 100%), so the Rule of 72 is not such a good approximation. The correct answer at 5% comes from solving $2x = x(1.05)^t$ using logarithms. We get $t \approx 14.2$, which is quite accurate.

   (c) They suggest that the Rule of 72 works well for small interest rates, but not for large ones.

6. (a) For Investment #1, use the annuity formula to get a present value of about \$772 at a 5% interest rate. For Investment #2, the brute force way to calculate the present value is to calculate the present value of each of the 6 lump sum payments and then add them up to get about \$835. A more elegant way is to note that Investment #2 is equivalent to a ten-year annuity *minus* a four-year annuity. You can therefore use the annuity formula to calculate the present value of the ten-year annuity (\$1,544) and the four-year annuity (\$709). Subtracting one from the other gives a present value for Investment #2 of \$835.

   Investment #2 is the better option at a 5% interest rate.

   (b) Following the same process described above, Investments #1 and #2 have present values of \$502 and \$433, respectively. So Investment #1

   (c) Higher interest rates favor Investment #1, in which you essentially forfeit money in the distant future in order to get more money in the immediate future. Since higher interest rates make the future less important, they make Investment #1 more attractive.

   (d) There are many real-life decisions with similar features; for example, Investment #1 could be going to work right out of high school, and Investment #2 could be going to college for 4 years first to increase your earnings potential.

7. (a) The king was supposed to deliver $2^{63} \approx 1,000,000,000,000,000,000,000$ grains of rice for the last square.

   (b) The pond was half-covered on the 29th day.

   (c) The answer is the height of a single sheet of paper multiplied by $2^{39} \approx 1,000,000,000,000$. If 1,000 pages makes an inch, then this gives us 1,000,000,000 inches, or about 83 million feet, or about 16,000 miles.

Comment: These examples involve interest rates of 100% (i.e., doubling), but you will get similar results with much smaller interest rates as long as your time horizons are long enough. This is because all interest rate problems share a common feature: constant doubling time. Put \$100 in the bank at 100% interest and it will double every year: \$200, \$400, \$800,... At 1% interest it will take 70 years to double to \$200, and every 70 years thereafter it will double again (to \$400, \$800,...). So if we call 70 years a *lifetime*, we see that an *annual* interest rate of 1% is equivalent to a *lifetime* interest rate of 100%. So 1% growth and 100% growth are different in degree but not in spirit.

8. (a) Using the annuity formula we get a present value of about \$6 trillion.
   (b) The expected damages are $\frac{1}{3}(6) + \frac{1}{3}(3) + \frac{1}{3}(0) \approx \$3$ trillion.
   (c) Plug \$3 trillion into the present value of a lump sum formula to get a present value of \$59 billion.
   (d) Using the present value of a lump sum formula, we get \$9,130.

9. (a) You're more likely to buy the TV set.
   (b) Increase.
   (c) More attractive.
   (d) Increase.
   (e) More attractive.
   (f) Up.

10. (a) Plug \$15 million, 200 years, and 2% into the future value of a lump sum formula to get a current bank account balance of \$787 million.
    (b) Plug \$15 million, 200 years, and 8% into the future value of a lump sum formula to get a current bank account balance of \$72.6 trillion

11. (a) The approximate annual interest rate is 6%.
    (b) Use the future value of a lump sum formula to calculate how much money we'll have at the end of 12 months if we put \$100 in the bank at a monthly interest rate of 0.5%, i.e. $s = 0.005$: FV= $\$100(1.005)^{12} \approx$ 106.17. So a monthly interest rate of 0.5% actually corresponds to an annual interest rate of 6.17%.
    (c) The approximate monthly interest rate is 0.5%.
    (d) Use the future value of a lump sum formula to set $x(1+s)^{12} = x(1.06)$ and solve for s using logarithms. This yields $s \approx .00487$, i.e., an interest rate of about 0.487%.

12. To calculate the present value of the yearly option, we need to split the payment into three parts: the \$500,000 you get immediately (which has a present value of \$500,000), the 28-year annuity (which, using the annuity formula, has a present value of about \$3.35 million), and the payment in year 29 (which, using the present value of a lump sum formula, has a present value of about \$460,400). We add these up to get a present value of about \$4.31 million.

    To calculate the present value of the weekly option, we first need to calculate a weekly interest rate that is in accordance with a yearly interest rate of 6%. A decent approximation is to divide the yearly interest rate by the number of weeks in a year (52), yielding a weekly interest rate of about 0.115%, i.e., $s \approx 0.00115$.[2] Using this value of s in the annuity formula along with \$5,000 and 2,000 weeks gives us a present value of about \$3.91 million for the weekly option.

    Finally, the present value of receiving \$3.5 million immediately is \$3.5 million. So the best option is the yearly payments.

Chapter 4: *Math*: Trees and Fish

1. These are all answered (to the best of my abilities) in the text.

Chapter 5: More Optimization over Time

1. The approximation is $14.4\% - 8\% = 6.4\%$. The actual answer comes from the formula, and gives a result of 5.9%.

2. This is explained in the text.

3. (a) Use the nominal interest rate and the future value formula to get a bank account balance of about \$259.37.
   (b) Use the inflation rate and the future value formula to get an apple fritter price of about \$1.63.
   (c) Today you have \$100 and fritters cost \$1, so you can buy $x = 100$ of them. In ten years you'll have \$259.37 and fritters will cost \$1.63, so you'll be able to buy about $y = 159$ of them. So we can calculate $z \approx 59$.
   (d) The rule of thumb approximation says that the real interest rate should be about $10\% - 5\% = 5\%$. The actual value is $\frac{1+.1}{1+.05} - 1 \approx .048$, i.e., 4.8%.

[2] If you don't like approximations, here's how to use logarithms to get the true value. If you put x in the bank at a weekly interest rate of $100 \cdot s\%$, after one year (52 weeks), you'll have $x(1+s)^{52}$ in the bank. We want this to equal $x(1.06)$, since this follows from a yearly interest rate of 6%. Setting $x(1+s)^{52} = x(1.06)$ and solving for s using logarithms gives us $s \approx .00112$, i.e., an interest rate of about 0.112%.

(e) If you put \$100 in the bank at this interest rate, after 10 years you'd have about \$159. So you get $z = 59$ as your gain in purchasing power.

4. **Use real when your payments are inflation-adjusted.**
   (a) Use the annuity formula and the real interest rate (about $6 - 4 = 2\%$) to get a present value of about \$470.
   (b) Use the perpetuity formula and the real interest rate (about $6 - 4 = 2\%$) to get a present value of about \$5,000.

   Use real to calculate future purchasing power.
   You're not figuring correctly because you're forgetting that prices are going to rise. Yes, you'll have 10 times more money, but you won't be able to buy 10 times more stuff. Using the real interest rate and the future value formula, we get a future value of \$11 million, or about 2.2 times more purchasing power.

5. Since we're trying to figure out what the current bank account balance is and the bank pays the nominal interest rate, we should use the nominal interest rate to determine if the Louisiana Purchase was really a great deal.

   Note that an estimate of 2% for the real interest rate actually does make sense; in fact, it's called the Fischer Hypothesis, which you might have (or may yet) come across in macroeconomics. The 8% figure for the nominal interest rate, in contrast, is entirely fictitious; you'll have to study some economic history if you want a real approximation for the nominal interest rate over the last 200 years.

6. We're trying to figure out how much money we need to put in the bank today in order to finance cash payments in the future. Since the bank pays the nominal interest rate, that's the rate we should use.

Chapter 6: Transition: Arbitrage

1. This is answered (to the best of my abilities) in the text.

2. If you lend the company \$100, the expected value of your investment is $.90(120) + .10(0) = \$108$, meaning that your expected rate of return is 8%. If government bonds have a 3% expected rate of return, the risk premium associated with the junk bonds is $8\% - 3\% = 5\%$.

3. (a) The *actual* rates of return turned out to be different, but it could still be true that at any point in time over the last twenty years the *expected* rates of return were the same. Economic reasoning suggests that they should have been; otherwise investors would not have been acting optimally. In terms of its effect on this logic, the fact that the actual rates of return turned out to be different is no more cause for concern than the fact that some lottery tickets turn out to be winners and some turn out to be losers.

(b) Again, actual rates of return can be different even though expected rates of return are the same at any moment in time. For example, there may be unexpected advances (or setbacks) in the development of electric cars or other alternatives to gasoline-powered cars.

An alternative explanation is that the cost of extracting oil has gone down over time; as a result, the profits from each barrel of oil may be going up at the rate of interest even though the price of oil may not steady or falling.

(c) Companies that issue junk bonds are financially troubled; if they go bankrupt, you lose both the interest and the principal. So even though junk bonds may offer 20% interest, their expected rate of return is much less than 20%, and therefore much closer to the expected rate of return of U.S. Treasury bonds. Also, the U.S. Treasury isn't likely to go bankrupt, meaning that two assets don't have comparable risk. So it is likely that there is a risk premium associated with the junk bonds.

(d) The answer here is that the rate of inflation is much higher in Turkey than in the U.S. So even though the two banks pay different nominal interest rates, their expected real interest rates may be equal.

One way to see the effect of inflation is to imagine that you'd invested 100 U.S. dollars in a Turkish bank in June 2000. The exchange rate then was about 610,000 Turkish lira to $1, so your $100 would have gotten you 61 million Turkish lira. After one year at a 59% interest rate, you would have 97 million lira. But when you try to change that back into U.S. dollars, you find that the exchange rate has changed: in June 2001 it's 1,240,000 lira to $1, so your 97 million lira buys only $78.23. You actually would have lost money on this investment!

Answer Key for Part II: One v. One, One v. Many

Chapter 7: Cake-Cutting

1. Arguably, differences in opinion make fair division easier, not harder. For example, if one child only likes vanilla and the other child only like chocolate, then cake division is, well, a piece of you-know-what.

2. This question is not fair game for exam, but is solvable via induction.

3. The Coase Theorem says that people who are free to trade have a strong incentive to trade until they exhaust all possible gains from trade, i.e., until they complete all possible Pareto improvements and therefore reach a Pareto efficient allocation of resources. The implication for the cake-cutting problem is that a "mom" whose sole concern is efficiency can divide the cake up however she wants—as long as the children can trade, they should be able to reach a Pareto efficient allocation regardless of the starting point. For example, if you give the chocolate piece to the kid who loves vanilla and the vanilla piece to the kid who loves chocolate, they can just trade pieces and will end up at a Pareto efficient allocation.

4. To show that envy-free implies proportional, we will show that not proportional implies not envy-free. If a cake division is not proportional, one child gets less than $\frac{1}{n}$th of the cake (in her estimation). This means that (according to her estimation) the other $(n-1)$ children get more than $\frac{n-1}{n}$th of the cake. Therefore at least one of the other children must have a piece bigger than $\frac{1}{n}$th, meaning that the cake division is not envy-free.

5. (a) Alice would spend 4 hours gathering veggies and 2 hours fishing, providing her with 4 veggies and 4 fish. Bob would do exactly the opposite (4 hours fishing, 2 hours gathering veggies) and would also end up with 4 of each.

   (b) If they specialize, Alice spends 6 hours fishing, so she gets 12 fish; Bob spends 6 hours hunting, so he gets 12 veggies. Then they split the results, so each gets 6 fish and 6 veggies, a clear Pareto improvement over the no-trade situation.

(c) Yes.

(d) They would allocate their time as before, but now Alice would get 12 fish and 12 veggies and Bob would get 4 fish and 4 veggies.7

(e) If they specialize as described in the problem, Alice ends up with 18 fish and 9 veggies, and Bob ends up with 1 fish and 10 veggies. After they trade, Alice ends up with 13 fish and 13 veggies, and Bob ends up with 6 fish and 6 veggies, another Pareto improvement!

(f) Yes.

(g) Alice must give up 2 fish to get one vegetable, and must give up 0.5 veggies to get one fish.

(h) Bob must give up 0.5 fish to get one vegetable, and 2 veggies to get one fish.

(i) Alice.

(j) Bob. When they concentrate on the items for which they are the least-cost producer, they can both benefit from trade even though Alice has an absolute advantage over Bob in both fishing and gathering veggies. This is the concept of **comparative advantage**.

Chapter 8: Economics and Social Welfare

1. (a) Pareto inefficient means that it's possible to make someone better off without making anyone else worse off; in other words, there's a "free lunch". Example: It is Pareto inefficient to give Tom all the chicken and Mary all the veggies because Tom's a vegetarian and Mary loves chicken.

   (b) A Pareto improvement is a reallocation of resources that makes one person better off without making anyone else worse off. Example: giving Tom the veggies and Mary the chicken is a Pareto improvement over giving Tom the chicken and Mary the veggies.

   (c) Pareto efficient means that there is no "free lunch", i.e., it's not possible to make someone better off without making anyone else worse off. Example: Giving Tom the veggies and Mary the chicken is a Pareto efficient allocation of resources.

2. (a) A Pareto efficient allocation of resources may not be good because of equity concerns or other considerations. For example, it would be Pareto efficient for Bill Gates to own everything (or for one kid to get the whole cake), but we might not find these to be very appealing resource allocations.

(b) A Pareto inefficient allocation is in some meaningful sense bad because it's possible to make someone better off without making anybody else worse off, so why not do it?

3. (a) Economists would use the concept of Pareto improvement to define "better". (Note that this does not include equity considerations, e.g., the idea that cutting the cake 50-50 might be "better" than giving one child the whole cake.)

   (b) The claim that any Pareto efficient allocation is a Pareto improvement over any Pareto inefficient allocation is not true. For example, giving one child the whole cake is a Pareto efficient allocation, and giving each child one-third of the cake and throwing the remaining third away is Pareto inefficient, but the former is not a Pareto improvement over the latter.

4. (a) No. One Pareto improvement would be for the publisher to provide me with the article for free: I'm better off, and they're not any worse off. Another Pareto improvement is for me to pay $.25 for the article: I'm not any worse off, and the publisher is better off by $.25.

   (b) Yes. Perfect price discrimination by a monopolist is Pareto efficient because it's not possible to make any of the customers better off without making the monopolist worse off.

   (c) The monopolist would need detailed information about all its different customers, e.g., the maximum amount each customer is willing to pay for each different article; such information is not readily available. There might also be a problem with resale, i.e., with some individuals paying others to purchase articles for them.

Chapter 9: Sequential Move Games

1. This is explained (to the best of my abilities) in the text. The basic idea is that you need to anticipate your rival's response.

2. Tipping at the beginning of the meal is problematic because then the waitress has no incentive to provide good service. (The tip is already sunk.) Tipping at the end of the meal is problematic because then the customer has no incentive to provide the tip. (The service is already sunk.)

3. (a) Backward induction predicts an outcome of (M: 35, PE: 5).

   (b) Yes.

   (a) Backward induction predicts an outcome of (M: 70, PE: 0).

   (b) No; a Pareto improvement is (M: 100, PE: 0).

4. (a) If there are 10 sticks on the table, you should be player 2. Whenever your opponent takes 1 stick, you take 2; when he takes 2 sticks, you take 1. So you can force your opponent to move with 7 sticks, then 4 sticks, then 1 stick—so you win!

   (b) Hint: The above answer suggests a general strategy to follow.

   (c) I only have a partial answer, so let me know if you solve all or part of this!

5. (a) With one period, Player 1 offers Player 2 a sliver, and Player 2 accepts. With two periods, Player 1 offers Player 2 half the cake, and Player 2 accepts. (Both know that if Player 2 refuses, half the cake melts, Player 2 will offer Player 1 a sliver of the remaining half, and Player 1 will accept.) With three periods, Player 1 offers Player 2 one-quarter of the cake, and Player 2 accepts. (Both know that if Player 2 refuses, she'll have to offer Player 1 at least half of the remaining half, meaning that she'll get at most one-quarter.)

   (b) With one period, Player 1 offers Player 2 a sliver, and Player 2 accepts. With two periods, Player 1 offers Player 2 two-thirds of the cake, and Player 2 accepts. (Both know that if Player 2 refuses, one-third of the cake melts, Player 2 will offer Player 1 a sliver of the remaining two-thirds, and Player 1 will accept.) With three periods, Player 1 offers Player 2 two-ninths of the cake, and Player 2 accepts. (Both know that if Player 2 refuses, she'll have to offer Player 1 at least two-thirds of the remaining two-thirds, meaning that she'll get at most two-ninths.)

   (c) This is the magic of the Coase Theorem. It is in neither player's interest to let the cake melt away, so they have a strong incentive to figure things out at the beginning and bring about a Pareto efficient outcome. You can see the same phenomenon at work in labor disputes and lawsuits, many of which get settled before the parties really begin to hurt each other.

6. I'd be happy to look over your work if you do this.

7. (a) The naive outcome is for X to choose C, Y to choose H, and Z to choose T, producing the "naive outcome" at the top of the game tree.

   (b) No. If X and Y choose C and H, Z will choose F because this produces a better outcome for Z: FT is better than TQ! (But now backward induction kicks in: Y anticipates this, and so Y will choose G instead of H—GH is better than HQ. But X anticipates this, and so knows that a choice of C will result in CQ. X then uses backward induction to solve the bottom half of the tree—Z will choose F in the top part and H in the lower part, so Y will choose H because HG is better than FG—and determine that a choice of T will result in TC. Since

X prefers TC to CQ, X chooses T in the first round, leading Y to choose H and Z to choose F.

(c) Backward induction leads to a result of (TC, HG, FQ).

(d) This is not Pareto efficient: the "naive" strategies produce better outcomes for all three teams!

(e) Statement #1 is false because each team's choice in the first round will have strategic implications for its options in the second round. Statement #2 is true because each team's choice in the second round has no further ramifications; since there are no more rounds, in the second round each team faces a simple decision tree.

(f) This is a time-consuming problem. Thanks to Kieran Barr for finding two strategies that yield this same outcome!

8. (a) The answer here depends on your assumptions. See below for my take on it...

   (b) The situation is Pareto inefficient.

   (c) The key issue here is that Butch Cassidy is a bank robber, and hence cannot be bound to contracts or other agreements. Sure, Harriman could pay him the money, but what guarantee does he have that this will make Butch stop robbing his train? A more likely outcome is that Butch will take the money and continue to rob the train, and then Harriman will be out even more money. So Harriman hires the superposse instead, even though both he and Butch would be better off with an alternative outcome.

   (d) Contracts can help by forcing players to act in certain ways; then the Coase them allows them to negotiate an efficient outcome. The Coase Theorem doesn't work in the case of Butch Cassidy because he's an outlaw: there's no way to bind an outlaw to an enforceable contract.

9. (a) If you make eye contact with the driver, the driver will pretend that she's not going to stop, and then you'll get scared and won't go for it, so then the driver *won't* stop.

   (b) The game tree here has you choosing to look or not look. If you choose not to look, the driver chooses to stop or not, and the payoffs are obvious. If you choose to look, the driver chooses to stop or not, and in each of those situations you must choose whether or not to push the issue.

10. (a) The game tree is pictured and described in the text.

    (b) As discussed in the text, backward induction predicts that Player 1 will immediately choose \$2 and end the game, yielding an outcome of $(2, 0)$.

    (c) No. There are many Pareto improvements, e.g., $(2, 2)$.

(d) You can do this with induction; this exercise also suggests why backward induction has the name it does.

(e) This is a truly difficult philosophical question. If you're interested, there's an interesting chapter (and a great bibliography) on this topic, in the guise of "the unexpected hanging", in Martin Gardner's 1991 book, *The Unexpected Hanging, and Other Mathematical Diversions.*

11. Well, the exam can't be on Friday, because then on Thursday night you'd think, "Aha! The exam's got to be Friday!" So then you wouldn't be surprised; so the exam can't be on Friday. But then the exam can't be on Thursday, because then on Wednesday night you'd think, "Aha! The exam can't be on Friday, so it's got to be Thursday!" So then you wouldn't be surprised; so the exam can't be on Thursday. But then the exam can't be on Wednesday, or Tuesday, or even Monday. An apparently noncontroversial statement by your teacher turns out to be quite treacherous!

Chapter 10: Simultaneous Move Games

1. (a) A good prediction is that everybody would drive to work because driving is a dominant strategy: no matter what everybody else does, you always get there 20 minutes faster by driving.

   (b) This outcome is not Pareto efficient because the commute takes 2 hours; a Pareto improvement would be for everybody to take the bus, in which case the commute would only take 40 minutes.

   (c) The central difficulty is *not* that you don't know what others are going to do; you have a dominant strategy, so the other players' strategies are irrelevant for determining your optimal strategy.

   (d) A reasonable mechanism might be passing a law that everybody has to take the bus or pay a large fine.

2. (a) A good prediction is that everybody will invest in the private good because it's a dominant strategy: no matter what everybody else does, you always get \$1 more by investing privately.

   (b) This outcome is not Pareto efficient because each player only gets a return of \$2; a Pareto improvement would be for everybody to invest in the public good, in which case each player would get a return of \$10.

   (c) The central difficulty is *not* that you don't know what others are going to do; you have a dominant strategy, so the other players' strategies are irrelevant for determining your optimal strategy.

   (d) A reasonable mechanism might be passing a law that everybody has to invest in the public good or pay a large fine.

3. (a) Playing D is a dominant strategy for each player, so we can expect an outcome of (D, D).
   (b) Using backward induction, we start at the end of the game, i.e., in the second round. Playing D is a dominant strategy for each player in this round, so we can expect an outcome of (D, D) in the second round. But the players will anticipate this outcome, so playing D becomes a dominant strategy in the first round too! As a result, the expected outcome is for both players to play D both times.

Chapter 11: Iterated Dominance and Nash Equilibrium

1. This is explained to the best of my abilities in the text. The key idea is to anticipate your opponent's behavior.

2. This is a hard philosophical problem. Show your work to me if you decide to tackle it.

3. U is not the best strategy for Player 1 if Player 2 plays R, M is not the best strategy for Player 1 if Player 2 plays C, and D is not the best strategy for Player 1 if Player 2 plays R. Similarly, there are no strictly dominant strategies for Player 2: L is not the best strategy for Player 2 if Player 1 plays U, C is not the best strategy for Player 2 if Player 1 plays U, and R is not the best strategy for Player 2 if Player 1 plays D.

4. (a) Here U is dominated by M for player 1, then C is dominated by L (or R) for player 2, then D is dominated by M for player 1, then R is dominated by L for player 2. The result: (M, L), with a payoff of (4,8). This is also the unique Nash equilibrium of the game; it is not a Pareto efficient outcome because of (D, R).
   (b) Here D is dominated by M for player 1, then C is dominated by L for player 2, then U is dominated by M for player 1, then L is dominated by R for player 2. The result: (M, R), with a payoff of (6,2). This is also the unique Nash equilibrium of the game; it is not a Pareto efficient outcome because of (U, C).
   (c) There are no strictly dominated strategies. The NE are (U, C), (D, L) and (M, R).
   (d) Here M is dominated by U for player 1, then L and R are dominated by C for player 2, then D is dominated by U for player 1. The result: (U, C), with a payoff of (5,4). This is also the unique Nash equilibrium of the game; it is a Pareto efficient outcome.
   (e) Here L is dominated by C for player 2, and that is as far as iterated dominance can take us. We do not get a unique prediction for the outcome of this game. (All that we can say is that a rational player

		Player 2		
		R	P	S
Player 1	R	0,0	-1,1	1,-1
	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

Table B.1: The payoff matrix for the game "Rock, Paper, Scissors"

2 would never play L.) With Nash, the NE are (M, C) and (D, R). Note that these Nash equilibria are a subset of the iterated dominance solutions; see the next problem for details.

5. (a) The payoff matrix is shown in Table B.1.
   (b) Iterated dominance does not help you solve this game because there are no dominated strategies.
   (c) In accordance with intuition, the NE is for both players to choose randomly among the three strategies.
6. Not fair game for the exam.
7.

		Entrant	
		Enter	Stay Out
Monopolist	War	10, -10	100, 0
	Peace	35, 5	100, 0

The Nash equilibria are (War, Stay Out) and (Peace, Enter). This suggests that backward induction is in fact a refinement or strengthening of Nash equilibrium (which it is, namely **subgame perfect Nash equilibrium**).

8. Not fair game for the exam.
9. Yes! As long as the interest rate is sufficiently low, the players can also cooperate by taking turns: (C, D), (D, C), (C, D),... Instead of gaining \$1 every stage (which is the result with the trigger strategies), each player now gains \$5 every two stages. As an exercise, you can formally define this strategy (what happens if the other player doesn't cooperate?) and determine the interest rates that allow this strategy as a Nash equilibrium and those that make this strategy a Pareto improvement over the trigger

strategy. There are also plenty of other strategies, e.g., tit-for-tat, that you can play around with if you wish.

10. One possibility is to proceed as follows: U is weakly dominated by M for Player 1, and then R is weakly dominated by L for Player 2, and then M is (strictly) dominated by D for Player 1, yielding a prediction of (D, L). But another possibility is to proceed like this: M is weakly dominated by D for Player 1, then L is weakly dominated by R for Player 2, then U is (strictly) dominated by D for Player 1, yielding a prediction of (D, R). Conclusion: the order of elimination matters for iterated weak dominance!

Calculus Problems

C-1. Player 1 chooses p to maximize

$$\begin{aligned} E(\pi_1) &= p[q(1) + (1-q)(0)] + (1-p)[q(0) + (1-q)(3)] \\ &= pq + (1-p)(1-q)(3). \end{aligned}$$

Similarly, player 2 chooses q to maximize

$$\begin{aligned} E(\pi_2) &= q[p(3) + (1-p)(0)] + (1-q)[p(0) + (1-p)(1)] \\ &= 3pq + (1-q)(1-p). \end{aligned}$$

Now, we want to find p and q that form a Nash equilibrium, i.e., that are mutual best responses. To do this, we take derivatives and set them equal to zero.

So: player 1 wants to choose p to maximize $E(\pi_1) = pq + 3(1-p)(1-q)$. Any value of p that maximizes this is either a corner solution (i.e., one of the pure strategies $p = 1$ or $p = 0$) or an interior solution with $0 < p < 1$, in which case the partial derivative of $E(\pi_1)$ with respect to p must be zero:

$$\frac{\partial E(\pi_1)}{\partial p} = 0 \Longrightarrow q - 3(1-q) = 0 \Longrightarrow 4q = 3 \Longrightarrow q = \frac{3}{4}.$$

This tells us that *any* interior value of p is a candidate maximum as long as $q = \frac{3}{4}$. Mathematically, this makes sense because if $q = \frac{3}{4}$ then player 1's expected payoff (no matter what his choice of p) is always

$$E(\pi_1) = pq + 3(1-p)(1-q) = \frac{3}{4}p + 3\frac{1}{4}(1-p) = \frac{3}{4}.$$

If player 2 chooses $q \neq \frac{3}{4}$ then player 1's best response is to choose $p = 1$ (if $q > \frac{3}{4}$) or $p = 0$ (if $q < \frac{3}{4}$).

We can now do the math for player 2 and come up with a similar conclusion. Player 2's expected payoff is $pi_2 = 3pq + (1-q)(1-p)$. Any value of q that maximizes this is either a corner solution (i.e., one of the pure strategies $q = 1$ or $q = 0$) or an interior solution with $0 < q < 1$, in which case

$$\frac{\partial E(\pi_2)}{\partial q} = 0 \Longrightarrow 3p - (1-p) = 0 \Longrightarrow 4p = 1 \Longrightarrow p = \frac{1}{4}.$$

So if player 1 chooses $p = \frac{1}{4}$ then any choice of q is a best response for player 2. But if player 1 chooses $p \neq \frac{1}{4}$ then player 2's best response is a pure strategy: if player 1 chooses $p > \frac{1}{4}$ then player 2's best response is to choose $q = 1$; if player 1 chooses $p < \frac{1}{4}$ then player 2's best response is to choose $q = 0$.

Now we can put our results together to find the Nash equilibria in this game. If player 1's choice of p is a best response to player 2's choice of q then either $p = 1$ or $p = 0$ or $q = \frac{3}{4}$ (in which case any p is a best response). And if player 2's choice of q is a best response to player 1's choice of p then either $q = 1$ or $q = 0$ or $p = \frac{1}{4}$ (in which case any q is a best response).

Three choices for player 1 and three choices for player 2 combine to give us nine candidate Nash equilibria:

Four pure strategy candidates : $(p = 1, q = 1), (p = 1, q = 0), (p = 0, q = 1), (p = 0, q = 0)$.

One mixed strategy candidate : $(0 < p < 1, 0 < q < 1)$.

Four pure/mixed combinations : $(p = 1, 0 < q < 1), (p = 0, 0 < q < 1), (0 < p < 1, q = 1), (0 < p < 1, q = 0)$.

We can see from the payoff matrix that there are two Nash equilibria among the four pure strategy candidates: $(p = 1, q = 1)$ and $(p = 0, q = 0)$. The other other two are not Nash equilibra. We can also see that the four pure/mixed combinations are not best responses; for example, $(p = 1, 0 < q < 1)$ is not a Nash equilibrium because if player 1 chooses $p = 1$ then player 2's best response is to choose $q = 1$, not $0 < q < 1$.

But the mixed strategy candidate does yield a Nash equilibrium: player 1's choice of $0 < p < 1$ is a best response as long as $q = \frac{3}{4}$. And player 2's choice of $0 < q < 1$ is a best response as long as $p = \frac{1}{4}$. So the player's strategies are mutual best responses if $(p = \frac{1}{4}, q = \frac{3}{4})$.

So this game has three Nash equilibria: two in pure strategies and one in mixed strategies.

C-2. Player 1 chooses p to maximize

$$\begin{aligned} E(\pi_1) &= p[q(0) + (1-q)(-1)] + (1-p)[q(-2) + (1-q)(1)] \\ &= p(q-1) + (1-p)(1-3q). \end{aligned}$$

Similarly, player 2 chooses q to maximize

$$\begin{aligned} E(\pi_2) &= q[p(0) + (1-p)(1)] + (1-q)[p(5) + (1-p)(-2)] \\ &= q(1-p) + (1-q)(7p-2). \end{aligned}$$

Now, we want to find p and q that form a Nash equilibrium, i.e., that are mutual best responses. To do this, we take derivatives and set them equal to zero.

So: player 1 wants to choose p to maximize $E(\pi_1) = p(q-1) + (1-p)(1-3q)$. Any value of p that maximizes this is either a corner solution (i.e., one of the pure strategies $p = 1$ or $p = 0$) or an interior solution with $0 < p < 1$, in which case the partial derivative of $E(\pi_1)$ with respect to p must be zero:

$$\frac{\partial E(\pi_1)}{\partial p} = 0 \Longrightarrow q - 1 - (1 - 3q) = 0 \Longrightarrow 4q = 2 \Longrightarrow q = \frac{1}{2}.$$

. This tells us that *any* interior value of p is a candidate maximum as long as $q = \frac{1}{2}$. Mathematically, this makes sense because if $q = \frac{1}{2}$ then player 1's expected payoff (no matter what his choice of p) is always

$$E(\pi_1) = p(q-1) + (1-p)(1-3q) = -\frac{1}{2}p + (1-p)\frac{-1}{2} = -\frac{1}{2}.$$

If player 2 chooses $q \neq \frac{1}{2}$ then player 1's best response is to choose $p = 1$ (if $q > \frac{1}{2}$) or $p = 0$ (if $q < \frac{1}{2}$).

We can now do the math for player 2 and come up with a similar conclusion. Player 2's expected payoff is $q(1-p) + (1-q)(7p-2)$. Any value of q that maximizes this is either a corner solution (i.e., one of the pure strategies $q = 1$ or $q = 0$) or an interior solution with $0 < q < 1$, in which case

$$\frac{\partial E(\pi_2)}{\partial q} = 0 \Longrightarrow 1 - p - (7p - 2) = 0 \Longrightarrow 8p = 3 \Longrightarrow p = \frac{3}{8}.$$

. So if player 1 chooses $p = \frac{3}{8}$ then any choice of q is a best response for player 2. But if player 1 chooses $p \neq \frac{3}{8}$ then player 2's best response is a pure strategy: if player 1 chooses $p > \frac{3}{8}$ then player 2's best response is to choose $q = 0$; if player 1 chooses $p < \frac{3}{8}$ then player 2's best response is to choose $q = 1$.

Now we can put our results together to find the Nash equilibria in this game. If player 1's choice of p is a best response to player 2's choice of q then either $p = 1$ or $p = 0$ or $q = \frac{1}{2}$ (in which case any p is a best response). And if player 2's choice of q is a best response to player 1's choice of p then either $q = 1$ or $q = 0$ or $p = \frac{3}{8}$ (in which case any q is a best response).

Three choices for player 1 and three choices for player 2 combine to give us nine candidate Nash equilibria:

Four pure strategy candidates : $(p = 1, q = 1), (p = 1, q = 0), (p = 0, q = 1), (p = 0, q = 0)$.

One mixed strategy candidate : $(0 < p < 1, 0 < q < 1)$.

Four pure/mixed combinations : $(p = 1, 0 < q < 1), (p = 0, 0 < q < 1), (0 < p < 1, q = 1), (0 < p < 1, q = 0)$.

We can see from the payoff matrix that there are no Nash equilibria among the four pure strategy candidates: We can also see that the four pure/mixed combinations are not best responses; for example, $(p = 1, 0 < q < 1)$ is not a Nash equilibrium because if player 1 chooses $p = 1$ then player 2's best response is to choose $q = 0$, not $0 < q < 1$.

But the mixed strategy candidate does yield a Nash equilibrium: player 1's choice of $0 < p < 1$ is a best response as long as $q = \frac{1}{2}$. And player 2's choice of $0 < q < 1$ is a best response as long as $p = \frac{3}{8}$. So the player's strategies are mutual best responses if $(p = \frac{3}{8}, q = \frac{1}{2})$.

So this game has one (mixed strategy) Nash equilibrium.

Chapter 12: Application: Auctions

1. (a) Auctions pit different suppliers against each other, and their individual incentives lead them to drive down the price. This helps ensure that Company X will not be paying much more for springs than it costs the suppliers to produce them.

   (b) The example in the ad above may not be as impressive as it sounds because of the potential for gaming: if Company X knows that a number of firms can produce the springs for about \$350,000, it has essentially nothing to lose by indicating a willingness-to-pay of \$500,000—or even \$1,000,000—because the auction dynamics will drive the price down toward \$350,000. An analogy may help: say I want to purchase a \$20 bill. As long as there are enough competitive bidders, I can more-or-less fearlessly say that I'm willing to pay up to \$1,000 for that \$20 bill; competitive pressures will force the winning bid down to about \$20.

2. (a) The intuition can be seen from an example: say you're willing to pay up to \$100, but you only bid \$90. Let y be the highest bid not including your bid. If $y < 90$ then you win the auction and pay y; in this case, bidding \$90 instead of \$100 doesn't help you or hurt you. If $y > 100$ then you lose the auction and would have lost even if you bid \$100; again, bidding \$90 instead of \$100 doesn't help you or hurt you. But if y is between \$90 and \$100 (say, $y = \$95$) then bidding \$90 instead of \$100 actively hurts you: you end up losing the auction when you would have liked to have won it. (You had a chance to get something you value at \$100 for a payment of only \$95, but you didn't take it.)

   (b) Again, the intuition can be seen in the same example in which you're willing to pay up to \$100. Assume that you bid \$110 and that y is the highest bid not including your bid. If $y < \$100$ then you win the auction and pay y; in this case bidding \$110 instead of \$100 doesn't help you or hurt you. If $y > \$110$ then you lose the auction; again, bidding \$110 instead of \$100 doesn't help you or hurt you. But if y is between \$100 and \$110 (say, $y = \$105$) then bidding \$110 instead of \$100 actively hurts you: you end up winning the auction when you would have liked to have lost it. (You pay \$105 for something you only value at \$100.)

3. You should bid less than your true value. If your true way is, say, \$100, then you are indifferent between having the object and having \$100. If you bid \$100, winning the auction won't make you better off; if you bid more than \$100, winning the auction will actually make you worse off. The only strategy that makes it possible for you to be better off is for you to bid less than \$100.

4. A reasonable response might start off by noting that bidders will behave differently in the two auctions: bidders will shade their bids in a first-price auction, but not in a second-price auction. So in a first-price auction you get the highest bid from among a set of relatively low bids, and in a second-price auction you get the second-highest bid from among a set of relatively high bids. It's no longer clear which auction has the higher payoff. (In fact, there is a deeper result in game theory, called the Revenue Equivalence Theorem, which predicts that both types of auctions will yield the same expected payoff.)

5. (a) There are two possible outcomes: either $\$x$ is the highest bid and you win the auction, or $\$x$ isn't the highest bid and you lose the auction.

   (b) Your expected value from bidding $\$x$ in the auction is

$$\begin{aligned}\text{EV(Bidding } \$x) &= \text{Pr(Your } \$x \text{ bid wins)} \cdot \text{Value(Winning)} \\ &\quad + \text{Pr(Your } \$x \text{ bid loses)} \cdot \text{Value(Losing)}.\end{aligned}$$

   Since the value of losing is zero (you get nothing, you pay nothing), the second term disappears. So your expected value boils down to something like

$$\text{EV(Bidding } \$x) = \text{Pr(Your } \$x \text{ bid wins)} \cdot \text{Value(Winning)}.$$

   (c) The expression above simplifies to

$$\text{EV(Bidding } \$x) = \text{Pr(Your } \$x \text{ bid wins)} \cdot (\text{Value of object} - \$x).$$

   Here we can see that bidding your true value is a bad idea: your expected value will never be greater than zero! We can also see the tension at work in first-price sealed bid auctions: by reducing your bid, you lower the probability that you will win, but you increase the value of winning. (Optimal bidding strategies in this case are complicated. How much to shade your bid is a difficult question, since it depends on how much you think other people will bid....)

(d) Your expected value of bidding $\$x$ reduces to

$$\text{EV(Bidding \$x)} = \Pr(\text{Your } \$x \text{ bid wins}) \cdot (\text{Value of object} - \$y)$$

where $\$y$ is the *second-highest* bid. Since the price you pay is not determined by your own bid, shading your bid below your true value doesn't help you. It only increases the probability that you will lose the bid when you would like to have won it. (The same is true for bidding over your true value. This only increases the probability that you will win the object and be forced to pay an amount greater than your true value.) You maximize your expected value by bidding your true value.

Chapter 13: Application: Marine Affairs

There are no problems in this chapter.

Chapter 14: Transition: Game Theory v. Price Theory

1. Ford, GM, Toyota, and a few other manufacturers dominate the market for new cars, so it is not a competitive market. In contrast, the market for used cars is pretty close to the competitive ideal. There are lots of small sellers—individuals looking to sell their cars—and lots of small buyers—individuals looking to buy a used car.

Answer Key for Part III: Many v. Many

Chapter 15: Supply and Demand: The Basics

1. This is the price at which the amount that buyers want to buy equals the amount that sellers want to sell. At a higher price, sellers want to sell more than buyers want to buy, creating incentives that push prices down toward the equilibrium. At a lower price, buyers want to buy more than sellers want to sell, creating incentives that push prices up toward the equilibrium.

2. (a) Demand increases. Equilibrium price up, quantity up.
   (b) Supply decreases. Equilibrium price up, quantity down.
   (c) Demand decreases. Equilibrium price down, quantity down.
   (d) Demand decreases. Equilibrium price down, quantity down.

3. (a) Demand increases. Equilibrium price up, quantity up.
   (b) Supply decreases. Equilibrium price up, quantity down.

4. (a) Demand increases. Equilibrium price up, quantity up.
   (b) Supply decreases. Equilibrium price up, quantity down.

5. (a) Supply increases. Equilibrium price down, quantity up.
   (b) Demand decreases. Equilibrium price down, quantity down.
   (c) Demand increases. Equilibrium price up, quantity up.
   (d) Demand increases. Equilibrium price up, quantity up.
   (e) Supply increases. Equilibrium price down, quantity up.

6. (a) Supply decreases. Equilibrium price up, quantity down.
   (b) Supply decreases. Equilibrium price up, quantity down.
   (c) Demand increases. Equilibrium price up, quantity up.
   (d) Demand increases. Equilibrium price up, quantity up.

7. (a) Supply decreases. Equilibrium price up, quantity down.
   (b) Supply decreases. Equilibrium price up, quantity down.
   (c) Demand decreases. Equilibrium price down, quantity down.
   (d) Demand decreases. Equilibrium price down, quantity down.
   (e) Demand decreases. Equilibrium price down, quantity down.
   (f) Supply increases. Equilibrium price down, quantity up.
   (g) Not necessarily. The equilibrium quantity would go up, but the equilibrium price would go down. The net impact on total expenditures pq is therefore unclear.

8. (a) Demand decreases. Equilibrium price down, quantity down.
   (b) Supply decreases. Equilibrium price up, quantity down.
   (c) Supply increases. Equilibrium price down, quantity up.
   (d) Demand decreases. Equilibrium price down, quantity down.
   (e) Demand decreases. Equilibrium price down, quantity down.

9. (a) Demand decreases. Equilibrium price down, quantity down.
   (b) Supply decreases. Equilibrium price up, quantity down.
   (c) Equilibrium quantity fell. The effect on equilibrium price is unclear, since the demand shift and the supply shift move in opposite directions.
   (d) A public scare would decrease demand for beef and increase demand for chicken. The equilibrium price and quantity of beef would therefore fall, and the equilibrium price and quantity of chicken would rise. These changes would benefit beef eaters and chicken suppliers and hurt chicken eaters and beef suppliers.

10. (a) Supply increases. Equilibrium price down, quantity up.
    (b) Demand decreases. Equilibrium price down, quantity down.
    (c) Demand decreases. Equilibrium price down, quantity down.
    (d) Demand decreases. Equilibrium price down, quantity down.
    (e) Yes.
    (f) No. Opening the Arctic National Wildlife Refuge would increase the equilibrium quantity.
    (g) No. Promoting substititutes to oil (e.g., coal and nuclear power) is a demand-side strategy.
    (h) Yes.

11. (a) Supply increases. Equilibrium price down, quantity up.
    (b) Lower.
    (c) Less attractive.
    (d) More oil.
    (e) Supply increases. Equilibrium price down, quantity up.

Chapter 16: Taxes

1. This is the logic identified in the text, e.g., "At a price of x with a \$.40 tax, buyers should be willing to buy exactly as much as they were willing to buy at a price of \$$(x + .40)$ without the tax." For per-unit taxes, you can also use the ideas of marginal cost and marginal benefit: a tax on the sellers increases marginal costs by the amount of the tax, and a tax on the buyers reduces marginal benefits by the amount of the tax. (Applying this marginal approach is a bit tricky for ad valorem taxes such as sales taxes. You need an additional assumption here about firm profits in equilibrium. . .)

2. (a) The equilibrium price is \$.80 per pound; the equilibrium quantity is 8 million pounds per day.

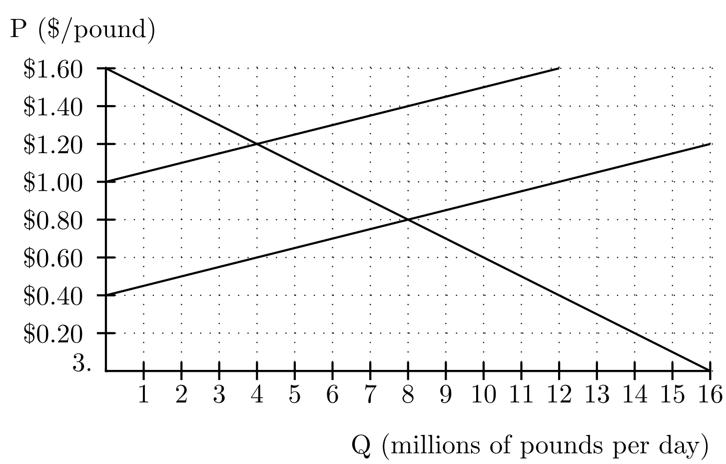
   (b) To find the slope of the supply curve, pick any two points on it—say, $(8, .80)$ and $(12, 1.00)$. Then the slope of the supply curve is

$$S_S = \frac{\text{rise}}{\text{run}} = \frac{1.00 - .80}{12 - 8} = \frac{.20}{4} = .05.$$

   Similarly, to find the slope of the demand curve, pick any two points on it—say $(8, .80)$ and $(12, .40)$. Then the slope of the demand curve is

$$S_D = \frac{\text{rise}}{\text{run}} = \frac{.40 - .80}{12 - 8} = \frac{-.40}{4} = -.1.$$

   So the ratio of the two slopes is $\left(\frac{S_D}{S_S}\right) = \frac{-.1}{.05} = -2$.



(a) The supply curve shifts up by \$.60.

(b) The new equilibrium features a price of \$1.20 per pound and a quantity of 4 million pounds per day.

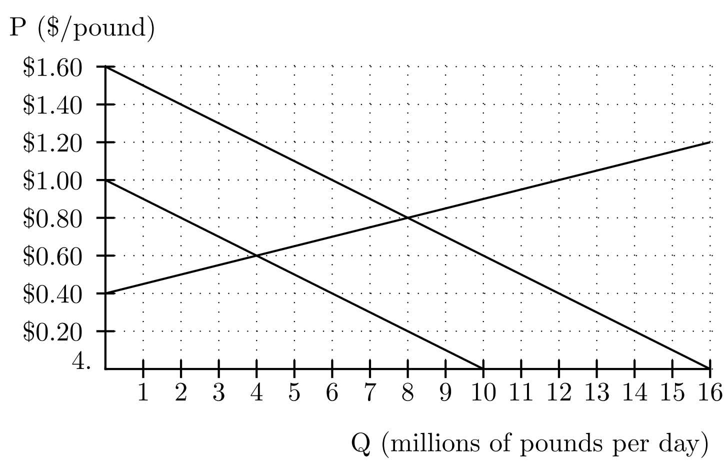
(c) A tax of \$.60 per pound levied on 4 million pounds per day yields revenue of (\$.60)(4) = \$2.4 million per day.

(d) The buyers pay \$1.20 for each pound of oranges.

(e) Before paying the \$.60 tax the sellers receive \$1.20 per pound, so after paying the tax the sellers receive \$.60 per pound.

(f) Without the tax, buyers paid \$.80 per pound; they now pay \$1.20 per pound, so they are worse off by $T_B = \$.40$ per pound. Similarly, the sellers received \$.80 per pound without the tax, but now they only receive \$.60 per pound, so they are worse off by $T_S = \$.20$ per pound. (As a check here, note that the \$.40 per pound impact on the buyers plus the \$.20 per pound impact on the sellers equals the \$.60 per pound tax.) The ratio of the tax burdens is $\frac{T_B}{T_S} = \frac{.40}{.20} = 2$.

(g) The tax burden ratio is the same magnitude as the ratio of the slopes calculated previously! Intuitively, this is because the ratio of the slopes measures the relative responsiveness of buyers and sellers to price changes. The side that is most responsive to price changes (in this case, the sellers) can push the lion's share of the tax burden onto the other side.



(a) The demand curve shifts down by \$.60.

(b) The new equilibrium features a price of \$.60 per pound and a quantity of 4 million pounds per day.

(c) A tax of \$.60 per pound levied on 4 million pounds per day yields revenue of (\$.60)(4) = \$2.4 million per day.

(d) The buyers pay \$.60 to the sellers for each pound of oranges, plus the tax of \$.60 to the government, so they pay a total of \$1.20 per pound.

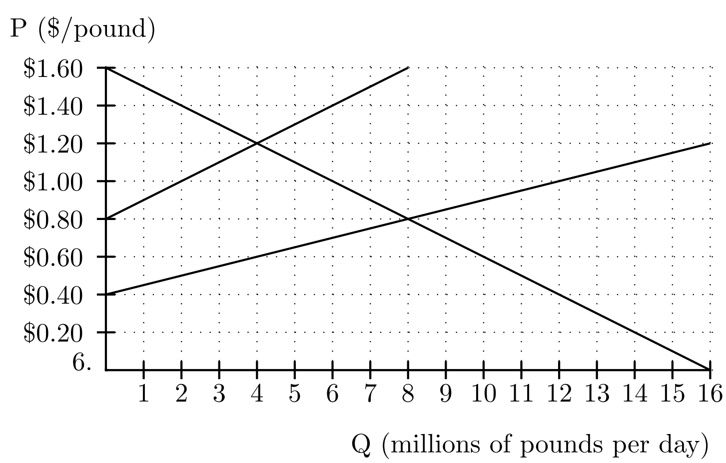
(e) The sellers receive \$.60 per pound of oranges.

(f) Without the tax, buyers paid \$.80 per pound; they now pay \$1.20 per pound, so they are worse off by $T_B = \$.40$ per pound. Similarly, the sellers received \$.80 per pound without the tax, but now they only receive \$.60 per pound, so they are worse off by $T_S = \$.20$ per pound. (As a check here, note that the \$.40 per pound impact on the buyers plus the \$.20 per pound impact on the sellers equals the \$.60 per pound tax.) The ratio of the tax burdens is $\frac{T_B}{T_S} = \frac{.40}{.20} = 2$.

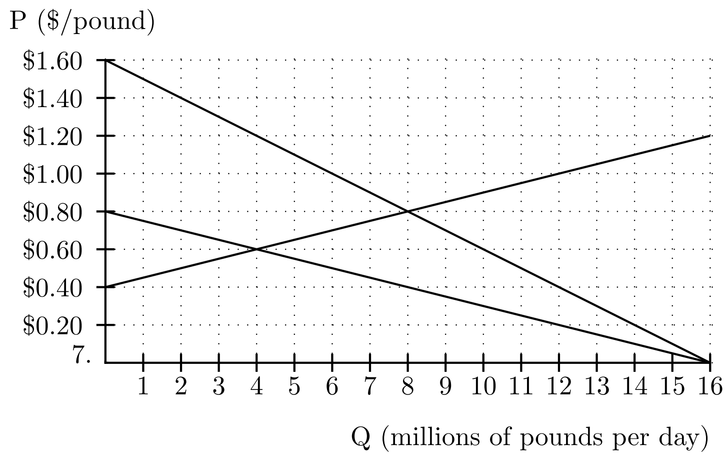
(g) The tax burden ratio is the same magnitude as the ratio of the slopes calculated previously!

5. The answers are essentially identical—regardless of the *legal* incidence of the tax (i.e., whether it's levied on the buyers or the sellers), the *economic* incidence of the tax comes out the same, i.e., the buyers always end up bearing \$.40 of the tax burden and the sellers end up bearing \$.20 of the tax burden. This is an example of the **tax incidence** result.

6.



The supply curve rotates as shown above. The remaining answers are identical to those above; note that at the equilibrium price of \$1.20 per pound, the 50% tax on the sellers amounts to \$.60 per pound.



The demand curve rotates as shown above. The remaining answers are identical to those above; note that at the equilibrium price of \$.60 per pound, the 100% tax on the buyers amounts to \$.60 per pound.

8. The answers are the same! Again, this is an example of tax equivalence, but with sales taxes the comparison is a little bit less obvious than with per-unit taxes: in the case of sales taxes, a 50% tax on the sellers is equivalent to a 100% tax on the buyers. This is because the market price is the price the buyer pays the seller; since the market price is twice as high when the tax is on the seller (\$1.20 versus \$.60), the sales tax rate on the sellers needs to be only half as high as the sales tax rate on the buyers in order to yield an equivalent result.

Algebra Problems

A-1. (a) Simultaneously solving the demand and supply equations gives us a market equilibrium of $p = \$.80$ per pound and $q = 8$ million pounds per day. Total revenue is therefore \$6.4 million per day.

(b) The slope of the demand curve is -10, so the price elasticity of demand at the market equilibrium is $-10\frac{.8}{8} = -1$. The slope of the supply curve is 20, so the price elasticity of supply at the market equilibrium is $20\frac{.8}{8} = 2$.

(c) A tax of \$.60 on the buyers will change the demand curve to $q = 16 - 10(p + .6)$, i.e., $q = 10 - 10p$. The supply curve is still $q = -8+20p$, so the new market equilibrium is at $p = \$.60$ per pound and $q = 4$ million pounds per day. The buyers end up paying $.60 + .60 = \$1.20$ per pound, and the sellers get \$.60 per pound. The original equilibrium was at a price of \$.80 per pound, so the buyers end up paying \$.40 more and the sellers end up getting \$.20 less. The ratio of these tax burdens is $\frac{.40}{.20} = 2$. This is the negative inverse of the ratio of the elasticities.

(d) A tax of \$.60 on the sellers will change the supply curve to $q = -8 + 20(p - .6)$, i.e., $q = -20 + 20p$. The demand curve is still $q = 16 - 10p$, so the new market equilibrium is at $p = \$1.20$ per pound and $q = 4$ million pounds per day. The buyers end up paying \$1.20 per pound, and the sellers get $1.20 - .60 = \$.60$ per pound. This is the same result as above, demonstrating the tax equivalence result.

(e) A tax of 50% on the sellers will change the supply curve to $q = -8+20(.5p)$, i.e., $q = -8+10p$. The demand curve is still $q = 16-10p$, so the new market equilibrium is at $p = \$1.20$ per pound and $q = 4$ million pounds per day. The buyers end up paying \$1.20 per pound, and the sellers get $.5(1.20) = \$.60$ per pound.

(f) A tax of 100% on the buyers will change the demand curve to $q = 16-10(2p)$, i.e., $q = 16-20p$. The supply curve is still $q = -8+20p$, so the new market equilibrium is at $p = \$.60$ per pound and $q = 4$ million pounds per day. The buyers end up paying $2(.60) = \$1.20$ per pound, and the sellers get \$.60 per pound. This the sales tax version of the tax equivalence result.

A-2. (a) The market demand curve is $q = 300(25 - 2p)$, i.e., $q = 7500 - 600p$. The market supply curve is $q = 500(5 + 3p)$, i.e., $q = 2500 + 1500p$.

(b) Solving simultaneously we get $p = \frac{50}{21} \approx= 2.38$ and $q \approx 6071$. Total revenue is therefore $pq \approx (2.38)(6071) \approx 14450$.

(c) The price elasticity of demand at the market equilibrium is given by

$$\varepsilon_D = \frac{dq}{dp}\frac{p}{q} = -600\frac{2.38}{6071} \approx -.235.$$

The price elasticity of supply at the market equilibrium is given by

$$\varepsilon_S = \frac{dq}{dp}\frac{p}{q} = 1500\frac{2.38}{6071} \approx .588.$$

The ratio of the elasticities is $\frac{-600}{1500} = -.4$.

(d) With a \$1 per-unit tax on the buyers, the market demand curve becomes $q = 7500 - 600(p + 1)$, i.e., $q = 6900 - 600p$. The market supply curve is still $q = 2500 + 1500p$, so the new equilibrium is at $p = \frac{44}{21} \approx 2.10$ and $q \approx 5643$. The seller therefore receives \$2.10 for each unit, and the buyer pays a total of $2.10 + 1.00 = \$3.10$ for each unit. Compared with the original equilibrium price of \$2.38, the seller is worse off by $2.38 - 2.10 = .28$, and the buyer is worse off by $3.10 - 2.38 = .72$. The ratio of these two is $\frac{.72}{.28} \approx 2.5$. Since $(.4)^{-1} = 2.5$, the tax burden ratio is the negative inverse of the ratio of the elasticities.

(e) With a \$1 per-unit tax on the sellers, the market supply curve becomes $q = 2500 + 1500(p - 1)$, i.e., $q = 1000 + 1500p$. The market demand curve is, as originally, $q = 7500 - 600p$. So the new equilibrium is at $p = \frac{65}{21} \approx \$3.10$ and $q \approx 5643$. This shows the tax equivalence result: the buyers and sellers end up in the same spot regardless of whether the tax is placed on the buyer or the seller.

(f) With a 50% sales tax on the sellers, the market supply curve becomes $q = 2500 + 1500(.5p)$, i.e., $q = 2500 + 750p$. The demand curve is, as originally, $q = 7500 - 600p$. So the new equilibrium is at $p = \frac{500}{135} \approx \$3.70$ and $q \approx 5278$.

(g) With a 100% sales tax on the buyers, the demand curve becomes $q = 7500 - 600(2p)$, i.e., $q = 7500 - 1200p$. The supply curve is, as originally, $q = 2500 + 1500p$. So the new equilibrium is at $p = \frac{50}{27} \approx \$1.85$ and $q \approx 5278$. This shows the tax equivalence result for sales taxes: the buyers and sellers end up in the same spot regardless of whether there's a 50% sales tax on the sellers or a 100% sales tax on the buyers.

Chapter 17: Elasticities

1. N/A

2. (a) Long-run supply curves are flatter, i.e., more elastic. This is because responsiveness to price changes increases in the long run. For example, in the long run sellers can build extra factories or close down existing factories when their useful lifetime expires.

   (b) Long-run demand curves are flatter, i.e., more elastic. Again, this is because responsiveness to price changes increases in the long run. For example, in the long run buyers of gasoline can respond to a price increase by buying a different car or moving where they live and/or where they work. Short-run options are more limited, leading to reduced responsiveness to price changes.

3. (a) The point Y corresponds to point A in the elasticity formula, so we have $p_A = \$.80$ and $q_A = 8$. For point B we can take any other point, e.g., the convenient point with $p_B = \$.60$ and $q_B = 10$. Then

$$\varepsilon = \frac{q_B - q_A}{p_B - p_A} \cdot \frac{p_A}{q_A} = \frac{2}{-.20} \cdot \frac{.80}{8} = -10 \cdot \frac{1}{10} = -1.$$

   (b) Point X is the equilibrium during bad years, when frost reduces supply.

(c) Total revenue at the three points are pq, i.e., $(1.2)(4) = \$4.8$ million per day at point X, $(.8)(8) = \$6.4$ million per day at point Y, and $(.2)(14) = \$2.8$ million per day at point Z.

(d) Growers make higher profits during "bad" years: their revenue is higher and their costs are assumed to be identical. This is basically a Prisoner's Dilemma situation for the growers: they would all be better off if they could restrict supply during "good" years, but the individual incentives lead them to flood the market and get low prices and low profits.

(e) The point Y corresponds to point A in the elasticity formula, so we have $p_A = \$.80$ and $q_A = 8$. For point B we can take any other point on the supply curve, e.g., the convenient point with $p_B = \$.60$ and $q_B = 4$. Then

$$\varepsilon = \frac{q_B - q_A}{p_B - p_A} \cdot \frac{p_A}{q_A} = \frac{-4}{-.20} \cdot \frac{.80}{8} = 20 \cdot \frac{1}{10} = 2.$$

So the ratio of the elasticities is $\dfrac{\epsilon_S}{\epsilon_D} = \dfrac{2}{-1} = -2$. This is the same as the ratio of the slopes calculated previously! (This result follows from problem 4.)

4.

$$\frac{\varepsilon_S}{\varepsilon_D} = \frac{\dfrac{\Delta q_S}{\Delta p_S}\dfrac{p}{q}}{\dfrac{\Delta q_D}{\Delta p_D}\dfrac{p}{q}} = \frac{\dfrac{\Delta p_D}{\Delta q_D}}{\dfrac{\Delta p_S}{\Delta q_S}} = \frac{S_D}{S_S}.$$

5. (a) At some market price p, firms making widgets earn the market rate of return; in the long run, then, firms are indifferent between making widgets and making other things, so they are willing to produce any number of widgets at price p. At any price less than p, firms would earn less than the market rate of return; in the long run, then, no firms would be willing to produce widgets, meaning that quantity supplied would be zero at any price less than p. Similarly, at any price greater than p, firms would earn more than the market rate of return; in the long run, then, everybody would rush into the widget-making business, meaning that the quantity supplied would be infinite at any price greater than p.

(b) A perfectly elastic supply curve for widgets is a horizontal line at some price p.

(c) A tax on the seller would shift the supply curve up by the amount of the tax. Since the supply curve is horizontal, the equilibrium price would increase by the full amount of the tax, meaning that buyers would pay the entire tax burden. (Similarly, a tax on the buyer would shift the demand curve down, but the equilibrium price would not change, meaning that the buyers bear the full burden of the tax.) This makes sense because of the analysis above: if sellers bear part of the tax burden then they would be earning less than the market rate of return. So in the long run buyers bear the entire burden of taxes in a competitive market.

Chapter 18: Supply and Demand: Some Details

1. Let the relevant variables be $p^H > p^L$ and $q^H > q^L$. A downward-sloping supply curve means q^L is optimal at the higher price (so that $p^H q^L - C(q^L)$ maximizes profits at price p^H) but that q^H is optimal at the lower price (so that $p^L q^H - C(q^H)$ maximizes profits at price p^L). To proceed by contradiction, note that profit maximization at the lower market price yields
$$p^L q^H - C(q^H) \geq p^L q^L - C(q^L).$$
It follows that q^L is not profit-maximizing at the higher price:
$$\begin{aligned} p^H q^H - C(q^H) &\geq (p^H - p^L)q^H + p^L q^L - C(q^L) \\ &= (p^H - p^L)(q^H - q^L) + p^H q^L - C(q^L) \\ &> p^H q^L - C(q^L). \end{aligned}$$

Chapter 19: Margins

There are no problems in this chapter.

Chapter 20: *Math:* Deriving Supply and Demand Curves

C-1. Microeconomics is about the actions and interactions of optimizing agents (e.g., profit-maximizing firms, utility-maximizing consumers). For differentiable functions with interior maxima or minima, the way to find those interior maxima or minima is to take a derivative and set it equal to zero. This gives you *candidate values* for maxima or minima; the reason is that slopes (i.e., derivatives) are equal to zero at the top of a hill (a maximum) or at the bottom of a valley (a minimum).

C-2. These are explained to the best of my abilities in the text.

C-3. (a) The equation of the indifference curve is $L^2 K = 9000$.

(b) We can rewrite the indifference curve as $K = 9000L^{-2}$, which has a slope of $\frac{dK}{dL} = -18000L^{-3}$.

(c) The marginal rate of substitute tells you how much cake the individual is willing to trade for one more latte. If $MRS = -3$, the firm should be willing to trade up to 3 pieces of cake to gain one latte; such a trade would leave the individual on the same indifference curve, i.e., allow the individual to reach the same utility level.

(d) The individual wants to choose L and K to minimize $C(L, K) = 2L + 3K$ subject to the utility constraint $L^2K = 9000$. The choice variables are L and K; the objective function is the cost function $C(L, K)$.

(e) Two equations in two unknowns usually yield a unique solution, so to solve this problem we will find two relevant equations involving L and K and solve them simultaneously. The first equation is the constraint, $U(L, K) = 9000$. The second equation comes from the last dollar rule: $\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}$.

(f) The marginal utility of lattes is $\frac{\partial U}{\partial L} = 2LK$ and the marginal utility of cake is $\frac{\partial f}{\partial K} = L^2$, so the last dollar rule gives us

$$\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K} \Longrightarrow \frac{2LK}{2} = \frac{L^2}{3},$$

which simplifies as

$$3LK = L^2 \Longrightarrow 3K = L.$$

Substituting this into our first equation, the constraint $L^2K = 9000$, yields $(3K)^2K = 9000$, which simplifies to $9K^3 = 9000$, and then to $K^3 = 1000$, and finally to $K = 10$. It follows from either of out equations that $L = 30$, so the minimum cost required to reach a utility level of 9000 is $p_L L + p_K K = 2(30) + 3(10) = 90$.

C-4. (a) The isoquant is $f(L, K) = q$, i.e., $L^{\frac{1}{4}}K^{\frac{1}{2}} = q$.

(b) Squaring both sides yields $L^{\frac{1}{2}}K = q^2$, i.e., $K = q^2L^{-\frac{1}{2}}$. The slope of this is $\frac{dK}{dL} = -\frac{1}{2}q^2L^{-\frac{3}{2}}$.

(c) The marginal rate of technical substitute tells you how much capital the firm is willing to trade for one more unit of labor. If $MRTS = -3$, the firm should be willing to trade up to 3 units of capital to gain one unit of labor; such a trade would leave the firm on the same isoquant, i.e., allow the firm to produce the same level of output.

(d) The firm wants to choose L and K to minimize $C(L, K) = 2L + 2K$ subject to $f(L, K) = q$. The choice variables are L and K; the objective function is the cost function $C(L, K)$.

(e) Two equations in two unknowns usually yield a unique solution, so to solve this problem we will find two relevant equations involving L and K and solve them simultaneously. The first equation is the constraint, $f(L,K)=q$. The second equation comes from the last dollar rule: $\frac{\frac{\partial f}{\partial L}}{p_L}=\frac{\frac{\partial f}{\partial K}}{p_K}$.

(f) The marginal product of labor is $\frac{\partial f}{\partial L}=\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{2}}$ and the marginal product of capital is $\frac{\partial f}{\partial K}=\frac{1}{2}L^{\frac{1}{4}}K^{-\frac{1}{2}}$, so the last dollar rule gives us

$$\frac{\frac{\partial f}{\partial L}}{p_L}=\frac{\frac{\partial f}{\partial K}}{p_K}\Longrightarrow\frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{2}}}{2}=\frac{\frac{1}{2}L^{\frac{1}{4}}K^{-\frac{1}{2}}}{2},$$

which simplifies as

$$\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{2}}=\frac{1}{2}L^{\frac{1}{4}}K^{-\frac{1}{2}}\Longrightarrow K=2L.$$

Substituting this into our first equation, the constraint $L^{\frac{1}{4}}K^{\frac{1}{2}}=q$, yields $L^{\frac{1}{4}}(2L)^{\frac{1}{2}}=q$, which simplifies to $L^{\frac{3}{4}}2^{\frac{1}{2}}=q$, and then to $L^{\frac{3}{4}}=q\cdot 2^{-\frac{1}{2}}$, and finally to $L=q^{\frac{4}{3}}2^{-\frac{2}{3}}$. Substituting this value of L into either of our two equations yields $K=q^{\frac{4}{3}}2^{\frac{1}{3}}$. So the minimum cost to produce 10 units of output is $p_LL+p_KK=2(q^{\frac{4}{3}}2^{-\frac{2}{3}})+2(q^{\frac{4}{3}}2^{\frac{1}{3}})=q^{\frac{4}{3}}(2^{\frac{1}{3}}+2^{\frac{4}{3}})\approx 3.78q^{\frac{4}{3}}$.

(g) Plugging in $q=10$ yields $L\approx 13.57$, $K\approx 27.14$, and a minimum cost of $p_LL+p_KK\approx 2(13.57)+2(27.14)=81.42$.

C-5. (a) The firm's job is to choose q to maximize profits $\pi=pq-C(q)=pq-3.78q^{\frac{4}{3}}$.

(b) Take a derivative with respect to the choice variable (q) and set it equal to zero. Cost minimization is a necessary condition for profit maximization because a firm that is producing q units of output in a non-cost-minimizing way can always increase profits by producing q units of output in the cost-minimizing way. So a profit-maximizing firm must be producing its optimal level of output at least cost.

(c) Taking a derivative of the profit function with respect to q and setting it equal to zero yields

$$\frac{d\pi}{dq}=0\Longrightarrow p-\frac{4}{3}(3.78)q^{\frac{1}{3}}=0\Longrightarrow p\approx 5.04q^{\frac{1}{3}}$$

Cubing both sides yields $p^3\approx 128q$, i.e., $q=\frac{p^3}{128}$. This is the firm's supply curve.

(d) Plugging $p=16$ into the supply curve yields $q\approx 32$. So its profits are $\pi=pq-C(q)\approx 16(32)-3.78(32)^{\frac{4}{3}}\approx 127.98$.

C-6. (a) The isoquant is $L^{\frac{1}{4}}(4)^{\frac{1}{2}}=q$, i.e., $L^{\frac{1}{4}}=\frac{q}{2}$, i.e., $L=\frac{q^4}{16}$. Note that the isoquant is not a line but *just a single point.* This is because capital is fixed at $K=4$, so the firm has no ability to trade-off between capital and labor.

(b) The firm wants to choose L to minimize $C(L,K) = 2L+2K = 2L+8$ subject to $f(L,K) = q$.

(c) Since the amount of capital the firm has is fixed, the firm cannot substitute between labor and capital. So the marginal rate of technical substitution is irrelevant in this problem.

(d) Capital is a sunk cost, so the price of capital will not affect the firm's behavior.

(e) In order to produce output of q, the firm has to hire $L = \frac{q^4}{16}$ units of labor. So the cost of producing q units of output is $C(q) = 2(\frac{q^4}{16}) + 2(4) = \frac{1}{8}q^4 + 8$.

(f) The firm wants to choose q to maximize profits $\pi = pq - C(q) = pq - (\frac{1}{8}q^4 + 8)$. To solve this problem we take a derivative with respect to q and set it equal to zero, yielding

$$\frac{d\pi}{dq} = 0 \Longrightarrow p - \frac{1}{2}q^3 = 0 \Longrightarrow 2p = q^3 \Longrightarrow q = (2p)^{\frac{1}{3}}.$$

C-7. (a) The profit function is $\pi = pq - C(q) = pq - 2q\sqrt{2}$. Taking a derivative and setting it equal to zero we get $p - 2\sqrt{2} = 0$, i.e., $p = 2\sqrt{2}$.

(b) At a price of $p = 2$, the firm will supply $q = 0$: its cost of producing each unit of output is $2\sqrt{2} > 2$, so it loses money on each unit it sells!

(c) At a price of $p = 4$, the firm will supply infinitely many units of output.

(d) We have $f(2L, 2K) = (2L)^{\frac{1}{2}}(2K)^{\frac{1}{2}} = 2L^{\frac{1}{2}}K^{\frac{1}{2}} = 2f(K,L)$.

(e) Since doubling inputs doubles output, the firm can double and redouble its profits simply by doubling and redoubling production (i.e., its choice of inputs). This (hopefully) helps explain why we get a corner solution (of $q = \infty$) when we attempt to maximize profits with $p = 4$.

C-8. (a) The individual wants to choose L and K to maximize utility $U(L,K) = L^2K$ subject to the budget constraint $p_L L + p_K K = M$.

(b) The solution method is to find two equations involving L and K and then solve them simultaneously. One equation is the constraint, $p_L L + p_K K = M$; the other is the last dollar rule, $\frac{\frac{\partial U}{\partial L}}{p_L} = \frac{\frac{\partial U}{\partial K}}{p_K}$.

(c) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $3K = L$ when $p_L = 2$ and $p_K = 3$. Substituting this into the budget constraint we have $2(3K) + 3K = 90$, i.e., $K = 10$. Substituting back into either of our equations yields $L = 30$.

(d) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $6K = p_L L$ when $p_K = 3$. Solving for K and substituting this into the budget constraint yields $p_L L + 3\left(\frac{1}{6} p_L L\right) = 90$, i.e., $1.5 p_L L = 90$. This simplifies to $L = \frac{60}{p_L}$, which is the Marshallian demand curve for lattes when $p_K = 3$ and $M = 90$. The slope of this demand curve is $\frac{dL}{dp_L} = -60 p_L^{-2}$; when $p_L = 2$, this simplifies to $\frac{-60}{4} = -15$.

(e) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $p_K K = L$ when $p_L = 2$. Using this to substitute for L in the budget constraint yields $2(p_K K) + p_K K = 90$, i.e., $K = \frac{30}{p_K}$, which is the Marshallian demand curve for cake when $p_L = 2$ and $M = 90$. The slope of this demand curve is $\frac{dK}{dp_K} = -30(p_K)^{-2}$; when $p_K = 3$, this simplifies to $-\frac{30}{9} \approx -3.33$.

(f) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $3K = L$ when $p_L = 2$ and $p_K = 3$. Using this to substitute for L in the budget constraint yields $2(3K) + 3K = M$, i.e., $K = \frac{M}{9}$, which is the Engel curve for cake when $p_L = 2$ and $p_K = 3$. Using the last dollar rule result $3K = L$ to substitute for K in the budget constraint yields $2L + L = M$, i.e., $L = \frac{M}{3}$, which is the Engel curve for cake when $p_L = 2$ and $p_K = 3$.

C-9. (a) Choose L and K to minimize $C(L, K) = p_L L + p_K K$ subject to the constraint $L^2 K = 9000$.

(b) Combine the constraint with the last dollar rule to get two equations in two unknowns. Solve these simultaneously to get the optimal values of L and K.

(c) We previously calculated the last dollar rule to yield $3K = L$ when $p_L = 2$ and $p_K = 3$. Substituting this into the constraint yields $(3K)^2 K = 9000$, which simplifies to $9K^3 = 9000$, i.e., $K = 10$. It follows from either equation that $L = 30$. The cost-minimizing cost is therefore $p_L L + p_K K = 2(30) + 3(10) = 90$.

(d) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $6K = p_L L$ when $p_K = 3$. Using this to substitute for K in the constraint $L^2 K = 9000$ yields $L^2 \frac{p_L L}{6} = 9000$, which simplifies to $L^3 = 54000 p_L^{-1}$, and then to $L = \left(54000 p_L^{-1}\right)^{\frac{1}{3}} = (30) 2^{\frac{1}{3}} p_L^{-\frac{1}{3}}$. The slope of this demand curve is $\frac{dL}{dp_L} = -\frac{1}{3}(30) 2^{\frac{1}{3}} p_L^{-\frac{4}{3}}$. When $p_L = 2$ this simplifies to $(-10) 2^{-1} = -5$.

(e) The last dollar rule gives us $\frac{2LK}{p_L} = \frac{L^2}{p_K}$, which simplifies to $p_K K = L$ when $p_L = 2$. Using this to substitute for L in the constraint $L^2 K = 9000$ yields $(p_K K)^2 K = 9000$, which simplifies to $p_K^2 K^3 = 9000$, and then to $K = \left(9000 p_K^{-2}\right)^{\frac{1}{3}} = (10) 3^{\frac{2}{3}} p_K^{-\frac{2}{3}}$. The slope of this demand curve is $\frac{dK}{dp_K} = -\frac{2}{3}(10) 3^{\frac{2}{3}} p_k^{-\frac{5}{3}}$. When $p_K = 3$ this simplifies to $-\frac{2}{3}(10) 3^{-1} = -\frac{20}{9} \approx -2.22$.

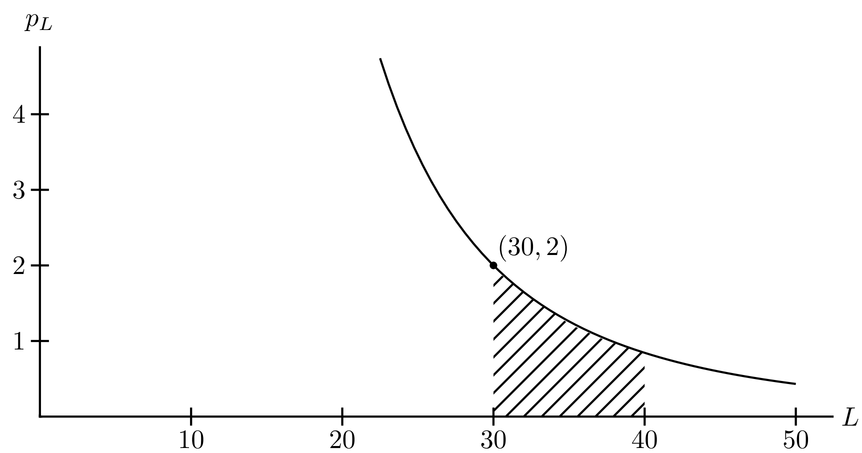


Figure B.3: The area under the *Hicksian* (utility-held-constant) demand curve measures willingness to pay.

C-10. The two problems are two sides of the same coin: if $U = 9000$ is the maximum utility that can be achieved with a budget of $M = 90$, then $M = 90$ is the minimum budget required to reach a utility level of $U = 9000$.

C-11. (a) What is the maximum amount of money this individual could exchange for 10 more lattes *and still be on the same indifference curve*, i.e., still have a utility level of 9000?

(b) We must have $L^2K = 9000$. So (1) if $L = 30$ then we need $K = 10$, which at a price of \$3 per cake requires a budget of \$30; (2) if $L = 40$ then we need $K = 5.625$, which at a price of \$3 per cake requires a budget of \$16.875; (3) subtracting yields \$30-\$16.875 = \$13.125 as this individual's willingness-to-pay for those 10 lattes.

(c) $p_L = 54000L^{-3}$

(d) See figure B.3.

(e) The area is the same as the willingness-to-pay calculated above!

$$\int_{30}^{40} 54000L^{-3}dL = -27000L^{-2}\Big|_{30}^{40} = -16.875 + 30 = \$13.125$$

(f) Various texts (e.g., Silberberg's *Structure of Economics*) insist that the area under Marshallian demand curves is meaningless.

Chapter 21: Transition: Welfare Economics

There are no problems in this chapter.

Answer Key for Appendixes

Appendix A: Government in Practice

There are no problems in this chapter.

Appendix B: *Math:* Monopoly and Oligopoly

1. Consider two firms, each with costs $C(q) = 3q^2$. They produce identical products, and the market demand curve is given by $q_1 + q_2 = 10 - p$. Find each firm's output and the market price under (1) collusion, (2) Cournot competition, and (3) Stackleberg leader-follower competition. Also find total industry profits under collusion.

   Firm 1's profits are

   $$\pi_1 = pq_1 - C(q_1) = (10 - q_1 - q_2)q_1 - 3q_1^2 = (10 - q_2)q_1 - 4q_1^2.$$

   Firm 2's profits are

   $$\pi_2 = pq_2 - C(q_2) = (10 - q_1 - q_2)q_2 - 3q_2^2 = (10 - q_1)q_2 - 4q_2^2.$$

   With collusion, the firms choose q_1 and q_2 to maximize joint profits

   $$\pi_1 + \pi_2 = (10 - q_2)q_1 - 4q_1^2 + (10 - q_1)q_2 - 4q_2^2.$$

   To solve this problem, we take partial derivatives with respect to each choice variable and set them equal to zero. This will give us two necessary first-order conditions (NFOCs) in two unknowns (q_1 and q_2); solving these simultaneously gives us our optimum.

   So: the NFOCs are

   $$\frac{\partial(\pi_1 + \pi_2)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 8q_1 - q_2 = 0$$

   and

   $$\frac{\partial(\pi_1 + \pi_2)}{\partial q_2} = 0 \Longrightarrow -q_1 + (10 - q_1) - 8q_2 = 0$$

Solving these jointly yields $q_1 = q_2 = 1$. The price is therefore $p = 10-2 = 8$ and industry profits are

$$\pi_1 + \pi_2 = 2\left(pq_1 - C(q_1)\right) = 2\left(8(1) - 3(1^2)\right) = 2(5) = 10.$$

Next, the Cournot problem. Here Firm 1 chooses q_1 to maximize its profits and Firm 2 chooses q_2 to maximize its profits. (The profit functions are given above.) To solve this problem we take a partial derivative of π_1 with respect to q_1 to get a necessary first-order condition (NFOC) for Firm 1. We then take a partial derivative of π_2 with respect to q_2 to get a necessary first-order condition (NFOC) for Firm 2. Solving these NFOCs simultaneously gives us the Cournot outcome.

So: the NFOCs are

$$\frac{\partial(\pi_1)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 8q_1 = 0$$

and

$$\frac{\partial(\pi_2)}{\partial q_2} = 0 \Longrightarrow (10 - q_1) - 8q_2 = 0$$

Solving these jointly yields $q_1 = q_2 = \frac{10}{9}$. The price is therefore $p = 10 - \frac{20}{9} = \frac{70}{9}$.

Finally, the Stackleberg problem. The objective functions look the same as in the Cournot case, but here we use backward induction to solve. Firm 2 sees what Firm 1 has chosen, and so chooses q_2 to maximize its profits. It does this by taking a partial derivative of π_2 (given above) with respect to q_2 to get an NFOC that is Firm 2's best response function to Firm 1's choice of q_1. Next, Firm 1 must anticipate Firm 2's reaction to its choice of q_1, and substitute this reaction function $q_2(q_1)$ into its profit function. Taking a partial derivative of the resulting profit function π_1 with respect to q_1 yields an NFOC that identifies Firm 1's profit-maximizing choice of q_1. Plugging this solution into Firm 2's best response function identifies Firm 2's profit-maximizing response of q_2.

So: The NFOC for Firm 2 is exactly as above: $10 - q_1 - 8q_2 = 0$, i.e., $q_2 = \frac{10-q_1}{8}$. Substituting this into Firm 1's profit function yields

$$\pi_1 = (10 - q_2)q_1 - 4q_1^2 = \left(10 - \frac{10 - q_1}{8}\right) q_1 - 4q_1^2 = \frac{1}{8}(70q_1 - 31q_1^2).$$

Taking a derivative with respect to q_1 gives us the NFOC

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \frac{1}{8}(70 - 62q_1) = 0 \Longrightarrow q_1 = \frac{35}{31} \approx 1.13.$$

Plugging this into Firm 2's best response function yields

$$q_2 = \frac{10 - q_1}{8} \approx \frac{10 - 1.13}{8} \approx 1.11.$$

2. The same problem, only now Firm 1 has costs of $C(q_1) = 4q_1^2$. (Firm 2's costs remain at $C(q_2) = 3q_2^2$.)

   Firm 1's profits are

   $$\pi_1 = pq_1 - C(q_1) = (10 - q_1 - q_2)q_1 - 4q_1^2 = (10 - q_2)q_1 - 5q_1^2.$$

   Firm 2's profits are, as above

   $$\pi_2 = pq_2 - C(q_2) = (10 - q_1 - q_2)q_2 - 3q_2^2 = (10 - q_1)q_2 - 4q_2^2.$$

   With collusion, the firms choose q_1 and q_2 to maximize joint profits

   $$\pi_1 + \pi_2 = (10 - q_2)q_1 - 5q_1^2 + (10 - q_1)q_2 - 4q_2^2.$$

   To solve this problem, we take partial derivatives with respect to each choice variable and set them equal to zero. This will give us two necessary first-order conditions (NFOCs) in two unknowns (q_1 and q_2); solving these simultaneously gives us our optimum.

   So: the NFOCs are

   $$\frac{\partial(\pi_1 + \pi_2)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 10q_1 - q_2 = 0$$

   and

   $$\frac{\partial(\pi_1 + \pi_2)}{\partial q_2} = 0 \Longrightarrow -q_1 + (10 - q_1) - 8q_2 = 0$$

   Solving these jointly yields $q_1 = \frac{15}{19} \approx .79$ and $q_2 = \frac{4}{3}q_1 \approx 1.05$. The price is therefore $p \approx 10 - .79 - 1.05 = 8.16$ and industry profits are

   $$\begin{aligned} \pi_1 + \pi_2 &\approx [(8.16)(.79) - 4(.79)^2] + [(8.16)(1.05) - 3(1.05)^2] \\ &\approx 3.95 + 5.26 = 9.21 \end{aligned}$$

   Next, the Cournot problem. Here Firm 1 chooses q_1 to maximize its profits and Firm 2 chooses q_2 to maximize its profits. (The profit functions are given above.) To solve this problem we take a partial derivative of π_1 with respect to q_1 to get a necessary first-order condition (NFOC) for Firm 1. We then take a partial derivative of π_2 with respect to q_2 to get a necessary first-order condition (NFOC) for Firm 2. Solving these NFOCs simultaneously gives us the Cournot outcome.

   So: the NFOCs are

   $$\frac{\partial(\pi_1)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 10q_1 = 0$$

   and

   $$\frac{\partial(\pi_2)}{\partial q_2} = 0 \Longrightarrow (10 - q_1) - 8q_2 = 0$$

Solving these jointly yields $q_1 = \frac{70}{79} \approx .89$ and $q_2 = \frac{90}{79} \approx 1.14$. The price is therefore $p \approx= 10 - .89 - 1.14 = 7.97$.

Finally, the Stackleberg problem. The objective functions look the same as in the Cournot case, but here we use backward induction to solve. Firm 2 sees what Firm 1 has chosen, and so chooses q_2 to maximize its profits. It does this by taking a partial derivative of π_2 (given above) with respect to q_2 to get an NFOC that is Firm 2's best response function to Firm 1's choice of q_1. Next, Firm 1 must anticipate Firm 2's reaction to its choice of q_1, and substitute this reaction function $q_2(q_1)$ into its profit function. Taking a partial derivative of the resulting profit function π_1 with respect to q_1 yields an NFOC that identifies Firm 1's profit-maximizing choice of q_1. Plugging this solution into Firm 2's best response function identifies Firm 2's profit-maximizing response of q_2.

So: The NFOC for Firm 2 is exactly as above: $10 - q_1 - 8q_2 = 0$, i.e., $q_2 = \frac{10-q_1}{8}$. Substituting this into Firm 1's profit function yields

$$\pi_1 = (10 - q_2)q_1 - 5q_1^2 = \left(10 - \frac{10 - q_1}{8}\right) q_1 - 5q_1^2 = \frac{1}{8}(70q_1 - 39q_1^2).$$

Taking a derivative with respect to q_1 gives us the NFOC

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \frac{1}{8}(70 - 78q_1) = 0 \Longrightarrow q_1 = \frac{70}{78} \approx .90.$$

Plugging this into Firm 2's best response function yields

$$q_2 = \frac{10 - q_1}{8} \approx \frac{10 - .90}{8} \approx 1.14.$$

3. The same problem, only now Firm 1 has costs of $C(q_1) = 3q_1^2$ and Firm 2 has costs of $C(q_2) = 4q_2^2$.

   The only difference between this problem and the previous problem is that Firm 1 and Firm 2 have switched places. So the solutions to the collusion and Cournot problems will be symmetric to the solutions above. Where the switch *is* important is in the Stackleberg game, because here the sequence of moves matters.

   So: the Stackleberg problem. The objective functions look the same as in the Cournot case, but here we use backward induction to solve. Firm 2 sees what Firm 1 has chosen, and so chooses q_2 to maximize its profits. It does this by taking a partial derivative of π_2 (given above) with respect to q_2 to get an NFOC that is Firm 2's best response function to Firm 1's choice of q_1. Next, Firm 1 must anticipate Firm 2's reaction to its choice of q_1, and substitute this reaction function $q_2(q_1)$ into its profit function. Taking a partial derivative of the resulting profit function π_1 with respect to q_1 yields an NFOC that identifies Firm 1's profit-maximizing choice of

q_1. Plugging this solution into Firm 2's best response function identifies Firm 2's profit-maximizing response of q_2.

So: The NFOC for Firm 2 is symmetric with Firm 1's best response function from the previous problem: $10 - q_1 - 10q_2 = 0$, i.e., $q_2 = \frac{10-q_1}{10}$. Substituting this into Firm 1's profit function yields

$$\pi_1 = (10 - q_2)q_1 - 4q_1^2 = \left(10 - \frac{10 - q_1}{10}\right) q_1 - 4q_1^2 = \frac{1}{10}(90q_1 - 39q_1^2).$$

Taking a derivative with respect to q_1 gives us the NFOC

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \frac{1}{10}(90 - 78q_1) = 0 \Longrightarrow q_1 = \frac{90}{78} \approx 1.15.$$

Plugging this into Firm 2's best response function yields

$$q_2 = \frac{10 - q_1}{10} \approx \frac{10 - 1.15}{10} \approx .89.$$

4. The same problem, with each firm having costs of $C(q) = 3q^2$, only now there is monopolistic competition, so that the inverse demand curve for Firm 1's output is $p_1 = 10 - 2q_1 - q_2$ and the inverse demand curve for Firm 2's output is $p_2 = 10 - q_1 - 2q_2$.

   Firm 1's profits are

$$\pi_1 = p_1 q_1 - C(q_1) = (10 - 2q_1 - q_2)q_1 - 3q_1^2 = (10 - q_2)q_1 - 5q_1^2.$$

   Firm 2's profits are

$$\pi_2 = p_2 q_2 - C(q_2) = (10 - q_1 - 2q_2)q_2 - 3q_2^2 = (10 - q_1)q_2 - 5q_2^2.$$

   With collusion, the firms choose q_1 and q_2 to maximize joint profits

$$\pi_1 + \pi_2 = (10 - q_2)q_1 - 5q_1^2 + (10 - q_1)q_2 - 5q_2^2.$$

   To solve this problem, we take partial derivatives with respect to each choice variable and set them equal to zero. This will give us two necessary first-order conditions (NFOCs) in two unknowns (q_1 and q_2); solving these simultaneously gives us our optimum.

   So: the NFOCs are

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 10q_1 - q_2 = 0$$

   and

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_2} = 0 \Longrightarrow -q_1 + (10 - q_1) - 10q_2 = 0$$

Solving these jointly yields $q_1 = q_2 = \frac{10}{12} \approx .83$. The prices are therefore $p_1 = p_2 \approx 10 - 3(.83) \approx 7.51$ and industry profits are

$$\pi_1 + \pi_2 = 2\left(p_1 q_1 - C(q_1)\right) \approx 2\left(7.51(.83) - 3(.83^2)\right) approx 4.17.$$

Next, the Cournot problem. Here Firm 1 chooses q_1 to maximize its profits and Firm 2 chooses q_2 to maximize its profits. (The profit functions are given above.) To solve this problem we take a partial derivative of π_1 with respect to q_1 to get a necessary first-order condition (NFOC) for Firm 1. We then take a partial derivative of π_2 with respect to q_2 to get a necessary first-order condition (NFOC) for Firm 2. Solving these NFOCs simultaneously gives us the Cournot outcome.

So: the NFOCs are

$$\frac{\partial(\pi_1)}{\partial q_1} = 0 \Longrightarrow 10 - q_2 - 10q_1 = 0$$

and

$$\frac{\partial(\pi_2)}{\partial q_2} = 0 \Longrightarrow (10 - q_1) - 10q_2 = 0$$

Solving these jointly yields $q_1 = q_2 = \frac{10}{11} \approx .91$. The prices are therefore $p_1 = p_2 \approx 10 - 3(.91) = 7.27$.

Finally, the Stackleberg problem. The objective functions look the same as in the Cournot case, but here we use backward induction to solve. Firm 2 sees what Firm 1 has chosen, and so chooses q_2 to maximize its profits. It does this by taking a partial derivative of π_2 (given above) with respect to q_2 to get an NFOC that is Firm 2's best response function to Firm 1's choice of q_1. Next, Firm 1 must anticipate Firm 2's reaction to its choice of q_1, and substitute this reaction function $q_2(q_1)$ into its profit function. Taking a partial derivative of the resulting profit function π_1 with respect to q_1 yields an NFOC that identifies Firm 1's profit-maximizing choice of q_1. Plugging this solution into Firm 2's best response function identifies Firm 2's profit-maximizing response of q_2.

So: The NFOC for Firm 2 is exactly as above: $10 - q_1 - 10q_2 = 0$, i.e., $q_2 = \frac{10-q_1}{10}$. Substituting this into Firm 1's profit function yields

$$\pi_1 = (10 - q_2)q_1 - 5q_1^2 = \left(10 - \frac{10 - q_1}{10}\right) q_1 - 5q_1^2 = \frac{1}{10}(90q_1 - 49q_1^2).$$

Taking a derivative with respect to q_1 gives us the NFOC

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \frac{1}{10}(90 - 98q_1) = 0 \Longrightarrow q_1 = \frac{90}{98} \approx .92.$$

Plugging this into Firm 2's best response function yields

$$q_2 = \frac{10 - q_1}{10} \approx \frac{10 - .92}{10} \approx .91.$$